# On the arithmetic properties of complex values of Hecke-Mahler series II.

The general case.

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**Abstract.** Here we characterise in a complete and explicit way the relations of algebraic dependence over $\mathbb{Q}$ of complex values of Hecke-Mahler series at algebraic points of the multiplicative group $\mathbb{G}_m^2$, providing a generalisation of the main Theorem in our work [Pel1]. Our main result, announced in [Pel2], contains previous Theorems by Loxton and van der Poorten, Mahler, and Masser.

## Contents



## 1 Introduction, results.

Let $w$ be a real positive irrational number. The *Hecke-Mahler series associated to $w$* is the power series:

$$f_w(u,v)=\sum_{l=1}^{\infty}\sum_{h=1}^{[lw]}u^lv^h,$$

where the square brackets denote the greatest integer part and sums with an empty set of terms are zero by convention. This transcendental series converges on the domain $\mathcal{D}_w$ whose elements are the couples of complex numbers $(u,v)$ such that $|u|<1$ and $|u||v|^w<1$. In [Mas2], Masser asked several questions which may all be included in the following Problem.

**Problem.** *Given an $m$-tuple $\mathcal{M} = ((u_1, v_1), \dots, (u_m, v_m)) = (\underline{u}_1, \dots, \underline{u}_m)$ of couples of non-zero complex numbers and positive irrationals $w_1, \dots, w_m$ with $\underline{u}_i \in \mathcal{D}_{w_i}$ for all $i$, compute the transcendence degree over $\mathbb{Q}$ of the subfield of $\mathbb{C}$:*

$$\mathbb{Q}(\underline{u}_1, \dots, \underline{u}_m, f_{w_1}(\underline{u}_1), \dots, f_{w_m}(\underline{u}_m)).$$

The main result of this text (Theorem 1) solves the particular case of this Problem with $w_1 = \cdots = w_m = w$ a quadratic irrational, and the couples of complex numbers $\underline{u}_i$ algebraic over $\mathbb{Q}$.

We introduce our main result. It will be helpful to do it first assuming the following natural condition on $w$ (see also condition (1.2) of [Pel1]). We suppose that $w = \theta$ satisfies:

$$0 < \theta < 1 \text{ and } \theta' < -1, \tag{1}$$

where the dash $'$ denotes the non-trivial automorphism of the quadratic number field $K := \mathbb{Q}(\theta)$. This condition, which says that $\theta^{-1}$ is reduced in the classical sense, can be easily dropped (see a few lines after the statement of our Theorem), and simplifies many proofs in this text.

We now need a few notations. Let us write $M = \mathbb{Z} + \theta^{-1}\mathbb{Z}$; it is a free $\mathbb{Z}$-submodule of $K$ of rank 2. Let $S$ be the stabiliser of $M$, that is, the order of $K$ defined by:

$$S := \{\beta \in K \text{ such that } \beta M \subset M\}.$$

Let $\beta$ be an element of $S$. There exist four rational integers $a, b, c, d$, uniques with the property that

$$\beta = c\theta^{-1} + d, \quad \beta\theta^{-1} = a\theta^{-1} + b. \tag{2}$$

The association $\beta \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines an algebraic action of $S$ on $\mathbb{G}_{\mathrm{m}}^2$:

$$(\underline{u} = (u, v) \in \mathbb{G}_{\mathrm{m}}^2, \beta \in S) \mapsto \underline{u}^\beta = (u^a v^b, u^c v^d), \tag{3}$$

where $a, b, c, d$ are determined by (2).

**Definition 1.1** The *saturated $S$-module* generated by $\underline{u} \in \mathbb{G}_{\mathrm{m}}^2$ is the set:

$$\Gamma_{\mathrm{sat}}(\underline{u}) := \{\underline{v} \in \mathbb{G}_{\mathrm{m}}^2 \text{ such that } \underline{v}^\beta = \underline{u}^\gamma, \text{ with } (\beta, \gamma) \in (S \setminus \{0\}) \times S\}.$$

Two elements $\underline{u}_1, \underline{u}_2 \in \mathbb{G}_{\mathrm{m}}^2$ are *$S$-equivalent* (we write $\underline{u}_1 \sim \underline{u}_2$) if

$$\Gamma_{\mathrm{sat}}(\underline{u}_1) = \Gamma_{\mathrm{sat}}(\underline{u}_2).$$

If $\underline{u}_1, \ldots, \underline{u}_m$ are points of $\mathbb{G}_{\mathrm{m}}^2$, the $S$-equivalence induces a partition

$$\mathcal{P} = \{\mathcal{I}_1, \ldots, \mathcal{I}_r\} \tag{4}$$

of the set $\{1, \ldots, m\}$ by the property: $i, j \in \mathcal{I}_s$ $(s = 1, \ldots, r)$ if and only if $\underline{u}_i \sim \underline{u}_j$. The main result of this text is the following.

**Theorem 1** *Let us consider $m$ algebraic points $\underline{u}_1, \ldots, \underline{u}_m \in \mathbb{G}_{\mathrm{m}}^2 \cap \mathcal{D}_\theta$ and the partition (4) of the set $\{1, \ldots, m\}$ associated to the $S$-equivalence. Let us also write:*

$$\mathcal{K} = \mathbb{Q}(f_\theta(\underline{u}_1), \ldots, f_\theta(\underline{u}_m)), \quad \mathcal{K}_s = \mathbb{Q}(f_\theta(\underline{u}_i))_{i \in \mathcal{I}_s}, \quad s = 1, \ldots, r.$$

*Let us denote by $d$ the transcendence degree of $\mathcal{K}$ and for all $s$, let us denote by $d_s$ the transcendence degree of $\mathcal{K}_s$. Then, we have*

$$d = d_1 + \cdots + d_r.$$

On can check (see section 2 in this text) that the points $\underline{u}_1, \ldots, \underline{u}_m$ are pairwise $S$-equivalent if and only if the $m$-tuple $(\underline{u}_1, \ldots, \underline{u}_m)$ satisfies the rank one hypothesis of [Pel1, section 2.3].

After Theorem 1, if $f_\theta(\underline{u}_1), \ldots, f_\theta(\underline{u}_m)$ are algebraically dependent, there exists a non-empty subset $\mathcal{I} \subset \{1, \ldots, m\}$ such that if $l$ is its cardinality, the $l$-tuple $(\underline{u}_i)_{i \in \mathcal{I}}$ satisfies the rank one hypothesis and the numbers $f_\theta(\underline{u}_i)$, $i \in \mathcal{I}$ are algebraically dependent. Hence, we can apply Theorem 1.1 of [Pel1] to characterise the $l$-tuple $(\underline{u}_i)_{i \in \mathcal{I}}$; in particular, we find that the numbers $f_\theta(\underline{u}_i), i \in \mathcal{I}$ and 1 are $\overline{\mathbb{Q}}$-linearly dependent. This explains why the present work (together with [Pel1]) solves the particular case of the Problem we are interested in.

We now explain how to get rid of the condition (1) in Theorem 1. We recall from [Pel1] the notation:

$$\mathcal{C}.\underline{u} = (u^a v^b, u^c v^d),$$

where $\mathcal{C} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with rational integer entries and $\underline{u} = (u, v)$ is a point of $\mathbb{G}_{\mathrm{m}}^2$. We may suppose that $0 < w < 1$: it has an ordinary continued fraction development

$$w = \frac{1}{d_0 +} \frac{1}{d_1 + \cdots} \frac{1}{b_0 +} \frac{1}{b_1 + \cdots} = [0, d_0, d_1, \ldots, d_g, \underbrace{b_0, b_1, \ldots, b_{2r-1}}_{\text{period}}, b_0, b_1, \ldots]$$

for $g \geq 0, r > 0$ and $d_0, \ldots, d_{g-1}, b_0, \ldots, b_{2r-1} \in \mathbb{Z}_{>0}$.

If $T = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix}$ is the matrix defined in (1.7) of [Pel1] and if $\theta$ is defined by $w = \frac{d\theta+b}{c\theta+a}$, then the transcendence degree of the fields

$$\mathbb{Q}(f_w(\underline{u}_1), \ldots, f_w(\underline{u}_m)) \quad \text{and} \quad \mathbb{Q}(f_\theta(T.\underline{u}_1), \ldots, f_\theta(T.\underline{u}_m))$$

are the same, after the arguments on p. 4 of [Pel1]. Hence, most of the time it suffices to work with $w = \theta$ satisfying (1). Notice however that the $m$-tuple $(T.\underline{u}_1, \ldots, T.\underline{u}_m)$ may not satisfy the rank one hypothesis even if the $m$-tuple $(\underline{u}_1, \ldots, \underline{u}_m)$ does.

From Theorem 1 and Theorem 1.1 of [Pel1] we can deduce the following Corollary.

**Corollary 1** *Let $H$ be a connected algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^2$ of dimension 1. Let $\underline{u}_1, \ldots, \underline{u}_m$ be algebraic elements of $H \cap \mathcal{D}_w$. Then, the complex numbers $f_w(\underline{u}_1), \ldots, f_w(\underline{u}_m)$ are algebraically independent over $\mathbb{Q}$ if and only if $\underline{u}_1, \ldots, \underline{u}_m$ are distinct.*

A proof of this Corollary, which holds for general quadratic irrational $w$'s, will be given in the case $w = \theta$ with $\theta$ satisfying (1) in section 4. Corollary 1 implies a Theorem of Masser in [Mas2] (by setting $H = \mathbb{G}_{\mathrm{m}} \times \{1\}$).

**Theorem 2 (Masser)** *Let $u_1, \ldots, u_m$ be algebraic numbers such that $0 < |u_i| < 1$ for all $i = 1, \ldots, m$. Then $f_w(u_1, 1), \ldots, f_w(u_m, 1)$ are algebraically independent if and only if $u_1, \ldots, u_m$ are distinct.*

Here is another consequence of our results; we recall from [Pel1] that a *torsion point $\underline{g}$* of $\mathbb{G}_{\mathrm{m}}^2$ is by definition, a couple of roots of unit.

**Corollary 2** *Let us suppose that $w = \theta$ satisfies (1). Let $\underline{u}_1, \ldots, \underline{u}_m$ be algebraic elements of $\mathbb{G}_{\mathrm{m}}^2 \cap \mathcal{D}_\theta$. Let us suppose that with the action (3), they generate an $S$-submodule of $\mathbb{G}_{\mathrm{m}}^2$ which does not contain any torsion point other than $(1,1)$. Then, we have that $\underline{u}_i^\eta \neq \underline{u}_j$ for all $1 \leq i < j \leq m$ and for all units $\eta$ of $S$, if and only if $f_\theta(\underline{u}_1), \ldots, f_\theta(\underline{u}_m)$ are algebraically independent over $\mathbb{Q}$.*

Corollary 2 implies a result of Loxton and van der Poorten in [Lo-Po2] (see Theorem 5 of this text) and a proof of it will also be included in section 4.

Our results imply that non-trivial relations between complex numbers such as
$$f_\theta(\underline{u}_1),\ldots,f_\theta(\underline{u}_m)$$
only arise if at least two of the points $\underline{u}_1,\ldots,\underline{u}_m$ are $S$-equivalent. Thus, we do not find new relations if we leave the rank one hypothesis of [Pel1]. This is also a typical feature of Fredholm series (see [Lo-Po3]).

Another analogy with Fredholm series is the fact that all the relations are linear, and can be very complicated. Some examples were given in [Pel1] (examples 1.3, 1.4 and 1.5). in section 5, some other families of "special" and "generic" relations will be explicitly described.

Here are a few words about the methods employed in this article.

As in [Pel1], the main difficulty we encounter is the fact that we must work with analytic functions of several complex variables allowing (at first sight) very complicate algebraic relations.

As in [Pel1], the use of the structure of $S$-module of $\mathbb{G}_\mathrm{m}^2$ given by (2) and (3) seems the only way to decrypt the relations between the numbers $f_w(\underline{u})$ and is a key point in this work.

In particular, we have the *functional equation*
$$f_\theta(\underline{u}^\eta)=f(\underline{u})-R(\underline{u}) \tag{5}$$
for $f_\theta$ (equation (1.11) of [Pel1]), valid for $\eta>1>\eta'>0$ a unit in $S$. By means of it, we are able to apply Mahler's method, thanks to several tools introduced in [Pel1].

We apply the same classical criterion of algebraic independence that we have used in [Pel1]: [Ni, Theorem 3.3.2, page 88]. Here however, we deal with functions of $2n$ complex variables with $n\geq 1$ (instead of 2 variables as in [Pel1]), and the matrix $\mathcal{C}(\eta)$ involved in the criterion above is no longer irreducible. Vanishing properties of certain locally analytic functions are more complicated than in [Pel1] (where we only needed Mahler's vanishing Lemma), and we must use the vanishing Theorem of Masser [Mas1] to check a technical condition, called "property A", playing the role of a zero estimate.

Masser's vanishing Theorem translates the analytic property A into a geometric property. This property can be very difficult to handle (this happens in several works by Loxton and van der Poorten; see for example Theorem 5 of this text).

Fortunately, in the situation of our text, this geometric property is equivalent to "multiplicative independence" of certain points, for our structure of

$S$-module. This is a second key point of our method making the "property A" transparent enough to conclude the proof.

Here is the plan of this text. After some preliminary tools for our proof (section 2, whose content will be described in its introduction) we will prove the main Theorem in section 3. In section 4 we will prove its Corollaries. In an Appendix, section 5, we will give some examples of relations between values of Hecke-Mahler series.

To end this section, we quote a result which follows from a combination of Theorem 1 and Theorem 1.1 of [Pel1], and which can be interesting as it gives a geometric interpretation of the relations between the values of $f_\theta$ at algebraic points of $\mathbb{G}_{\mathrm{m}}^2$.

**Theorem 3** *Let $m \geq 1$ be an integer. There exists a finite set $\mathcal{V}$ of proper algebraic subgroups of even dimension of $\mathbb{G}_{\mathrm{m}}^{2m}$, with the following property. Let $\mathcal{M} = (\underline{u}_1, \ldots, \underline{u}_m)$ be an algebraic point of $\mathbb{G}_{\mathrm{m}}^{2m}$. If for all $m$-tuples $(\eta_1, \ldots, \eta_m)$ of units of $K$ we have:*

$$(\underline{u}_1^{\eta_1}, \ldots, \underline{u}_m^{\eta_m}) \notin \bigcup_{\mathcal{Z} \in \mathcal{V}} \mathcal{Z},$$

*Then the complex numbers $f_\theta(\underline{u}_1), \ldots, f_\theta(\underline{u}_m)$ are algebraically independent over $\mathbb{Q}$.*

The proof of this result follows from the above mentioned results in a rather elementary way; we do not give more details here. It is also possible to sharpen this statement get a necessary and sufficient condition for algebraic independence.

# 2 Preliminaries to the proof of our Theorem.

In this section wee recall the main notations from [Pel1] and we introduce some new tools for our proof. The reader will find: a criterion of irrationality for functions (Lemma 2.2), an account of the basic properties of the multiplicative independence in $\mathbb{T}$ (Lemma 2.3), a connection of this property with the "property A" of Loxton and van der Poorten (Lemma 2.4) and a result which allows to construct "positive" $S$-bases in finitely generated $S$-submodules of $\mathbb{T}$ (Proposition 1).

In the following, we will denote by $M^*$ the dual (for the trace $\mathbf{t}$) of a free $\mathbb{Z}$-module $M$ of rank 2 in $K$. We recall that if $M \subset N$ are two free

$\mathbb{Z}$-submodules of $K$ of rank 2, then $N^* \subset M^*$. Moreover, $S(M) = S(M^*)$ and $S(M)$ is itself a free $\mathbb{Z}$-module of rank 2.

From now on, we assume that $w = \theta$ is a positive quadratic irrational satisfying (1), and we write

$$f(\underline{u}) := f_\theta(\underline{u})$$

(unless otherwise specified).

We choose the basis $(B_0, B_1) = (\theta^{-1}, 1)$ of the $\mathbb{Z}$-module $M = \mathbb{Z} + \theta^{-1}\mathbb{Z}$. We also write:

$$\mathfrak{B} = \begin{pmatrix} B_0^* & B_0^{*\prime} \\ B_1^* & B_1^{*\prime} \end{pmatrix} = \delta^{-1} \begin{pmatrix} 1 & -1 \\ -\theta'^{-1} & \theta^{-1} \end{pmatrix},$$

where

$$\delta = \theta^{-1} - \theta'^{-1},$$

so that $(B_0^*, B_1^*)$ is the basis of $M^*$ dual of $(B_0, B_1)$.

We consider the exponential function (with $e(\tau) = e(2\pi \mathrm{i} \tau)$, $e(a_1, \ldots, a_s) = (e(a_1), \ldots, e(a_s))$, $\mathrm{i} := \sqrt{-1}$):

$$\Phi(z, z') = {}^{\mathrm{t}}e(\mathfrak{B} \cdot {}^{\mathrm{t}}(z, z')). \tag{6}$$

We will also denote by $\mathcal{D} = \mathcal{D}_\theta$ the domain of convergence of $f$. At the same time, we need to work with another domain, $\mathcal{D}^+$, which is the a domain of convergence of the twin series $f^+$ defined in section 3.1 of [Pel1]. If $p, q$ are two coprime positive integers such that $2/\mathbf{t}(\theta^{-1}) = p/q$, then

$$\mathcal{D}^+ = \{(u, v) \in \mathbb{C}^2 \text{ such that } |u| < 1 \text{ and } |u|^p |v|^q < 1\}.$$

If $\mathcal{H}$ is the complex upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \text{ such that } \Im(z) > 0\},$$

($\Im(\cdot)$ denotes the imaginary part of a complex number) and if

$$\mathcal{W}^+ \quad := \quad \{(z, z') \in \mathcal{H} \times \mathbb{C} \text{ with } -\Im(z) < \Im(z') < \Im(z)\},$$

then $\mathcal{D}^+ \cap \mathbb{T} = \Phi(\mathcal{W}^+)$ where

$$\mathbb{T} := \mathbb{G}_{\mathrm{m}}^2$$

(see equality (4.11) of [Pel1]).

In fact, $\mathcal{D}$ can be described in a similar way by setting

$$\mathcal{W} \quad := \quad \{(z, z') \in \mathcal{H} \times \mathbb{C} \text{ with } \Im(z') < \Im(z)\}.$$

One shows (cf. loc. cit.) that $\mathcal{D}\cap\mathbb{T}=\Phi(\mathcal{W})$, from which one sees that $\mathcal{D}^+\subset\mathcal{D}$.

In the following, lower case greek letters will exclusively denote elements of $K$. Lower case italic letters will often denote complex numbers; they can occasionally be in $K$. The notations $\underline{z},\underline{a},\underline{x},\ldots$ will indicate $l$-tuples (with $l>1$) of complex numbers. If $l=2$, the couples will be of the form $(z,z'),(a,a'),(x,x'),\ldots$ but if $x$ (or $x'$) does not belong to $K$, the dash $'$ will have nothing to do with the Galois conjugation. If $\beta$ is in $K$ and $\underline{z}=(z,z')\in\mathbb{C}^2$, we will write $\beta\underline{z}=(\beta z,\beta' z')$.

The exponential function $\Phi$ induces an action of the stabiliser $S=S(M)$ of $M$ over $\mathbb{T}$ defined by:

$$\begin{array}{ccc} S\times\mathbb{T} & \to & \mathbb{T} \\ (\beta,\underline{u}) & \mapsto & \underline{u}^\beta:=\Phi(\beta\underline{z}), \end{array}$$

where $\underline{z}=(z,z')\in\mathbb{C}^2$ is such that $\Phi(\underline{z})=\underline{u}$.

For $\nu\in K^\times$, let $\mathcal{B}(\nu)$ be the matrix defined in (2.4) of [Pel1]:

$$\mathcal{B}(\nu) \quad = \quad \mathfrak{B}\cdot\begin{pmatrix}\nu & 0\\ 0 & \nu'\end{pmatrix}\cdot\mathfrak{B}^{-1}. \tag{7}$$

It is easy to check that if $\beta\in S$, then $\mathcal{B}(\beta)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $a,b,c,d\in\mathbb{Z}$, and:

$$\underline{u}^\beta=\mathcal{B}(\beta).\underline{u}:=(u^av^b,u^cv^d).$$

This is the action introduced in (2) and (3).

By Lemma 2.1 of [Pel1], a point $\underline{g}\in\mathbb{T}$ is torsion if and only if there exists $\alpha\in K$ such that $\Phi(\alpha\underline{1})=\underline{g}$ (where $\underline{1}=(1,1)$). We remark that if $\underline{u},\underline{v}\in\mathbb{T}$ are non-torsion, then only two cases can occur.

1. We have $\underline{u}\sim\underline{v}$ and
$$\underline{u}^\gamma=\Phi(\alpha\underline{1})\underline{v}^\beta, \tag{8}$$
for some $\alpha\in K$ and $\beta,\gamma\in S\setminus\{0\}$.

2. We have $\underline{u}\not\sim\underline{v}$ and for all $\beta,\gamma\in S$, not both zero, the point $\underline{u}^\beta\underline{v}^\gamma$ is non-torsion.

If (8) holds, there exists $\underline{w}\in\mathbb{T}$, such that $\underline{w}^\gamma=\underline{v}$, with $\gamma$ as in (8). After Lemma 2.1 of [Pel1], we can find $\delta\in K,\rho\in S\setminus\{0\}$ such that $\underline{u}=\Phi(\delta\underline{1})\underline{w}^\rho$.

Let us consider non-torsion points $\underline{u}_1,\ldots,\underline{u}_m\in\mathbb{T}$; the relation $\sim$ determines a partition $(\mathcal{I}_s)_{s=1,\ldots,r}$ of the set $\{1,\ldots,m\}$. A little induction and the argument above show that for all $s = 1, \dots, r$, there exists a non-torsion point $\underline{v}_s \in \mathbb{T}$ such that for all $i \in \mathcal{I}_s$, there exists $\alpha_i \in K$ and $\beta_i \in S \setminus \{0\}$ satisfying:

$$\underline{u}_i = \underline{g}_i \underline{v}_s^{\beta_i}, \tag{9}$$

with $\underline{g}_i = \Phi(\alpha_i \underline{1})$ torsion point. This shows that for all $s = 1, \dots, r$, the collection

$$(\underline{u}_i)_{i \in \mathcal{I}_s}$$

satisfies the rank one hypothesis defined in section 2.3 of [Pel1].

## Irrationality of certain series.

**Definition 2.1** A *strictly convex cone* $\nabla$ of $\mathbb{R}^n$ of *axis* $\underline{Y} \in \mathbb{R}^n \setminus \{\underline{0}\}$ is, when $n > 1$, the union of all the half-lines $\underline{Z}\mathbb{R}_{>0}$ with $\underline{Z} \neq \underline{0}$ in $\mathbb{R}^n$, such that the plane angle between $\underline{Y}$ and $\underline{Z}$ has absolute value less than, or equal to $q$, for some $q < \pi/2$. When $n = 1$ by definition, there are two strictly convex cones only; the positive and the negative half-lines of $\mathbb{R}$. When $n > 1$, the supremum of the numbers $q$ is called the *amplitude* of $\nabla$, and the point $\underline{0} = (0, \dots, 0)$ its *apex*.

The lemma below is quoted for later use, and collects some elementary properties of cones.

**Lemma 2.1** *Let $\nabla$ be a strictly convex cone in $\mathbb{R}_{>0} \times \mathbb{R}^{n-1}$, let $\underline{y} = (y_1, y_2, \dots, y_n)$ be a point of $\mathbb{R}^n$.*

1. *If $n > 1$, the intersection between $\nabla$ and a plane $\mathcal{P}$ through $\underline{y}$ and the line determined by the axis of $\nabla$ is a strictly convex cone of $\mathbb{R}_{>0} \times \mathbb{R}$.*
2. *If $y_1 < 0$ then the distance between $\underline{y}$ and $\nabla$ (that is, the infimum of the distance between $\underline{y}$ and a point of $\nabla$) is strictly positive.*
3. *If $y_1 = 0$, then the distance between $\underline{y}$ and $\nabla$ is zero if and only if $\underline{y} = \underline{0}$ (that is, if and only if $\underline{y}$ is the apex of $\nabla$).*

**Proof.** The first property is a direct consequence of the definition of a strictly convex cone. The second and the third properties follow from the first by elementary plane geometry in the case $n > 1$, and are trivial in the case $n = 1$. $\square$

We have the isomorphism of $\mathbb{Z}$-modules

$$\ell : \mathbb{Z}^{2n} \to (M^*)^n$$

defined by:

$$\ell(p_1, q_1, \dots, p_n, q_n) = (\gamma_1, \dots, \gamma_n), \tag{10}$$

where for $i = 1, \dots, n$:

$$\gamma_i = \delta^{-1}(p_i - q_i\theta'^{-1}) \in M^*.$$

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ be an element of $(M^*)^n$. We write:

$$M(\underline{V})^{\underline{\lambda}} = u_1^{p_1} v_1^{q_1} \cdots u_n^{p_n} v_n^{q_n}, \tag{11}$$

where $p_i, q_i$ $(i = 1, \dots, n)$ are integers such that $\ell(p_1, q_1, \dots, p_n, q_n) = \underline{\lambda}$.

We denote by $\mathbb{C}\{\underline{V}\} = \mathbb{C}\{\underline{v}_1, \dots, \underline{v}_n\}$ the set of the formal Laurent series in $2n$ variables with complex coefficients, which is a $\mathbb{C}$-vector space of infinite dimension. A series $Q$ in $\mathbb{C}\{\underline{V}\}$ is of the form:

$$Q(\underline{V}) = Q(u_1, v_1, \dots, u_n, v_n) = \sum_{(p_1, q_1, \dots, p_n, q_n) \in \mathbb{Z}^{2n}} c_{p_1, q_1, \dots, p_n, q_n} u_1^{p_1} v_1^{q_1} \cdots u_n^{p_n} v_n^{q_n},$$

with $c_{p_1, q_1, \dots, p_n, q_n} \in \mathbb{C}$ for all $(p_1, q_1, \dots, p_n, q_n) \in \mathbb{Z}^{2n}$. This vector space contains the integral domain of series which are convergent in a neighbourhood of $\underline{0} = (0, \dots, , 0)$.

Let $Q(\underline{V}) \in \mathbb{C}\{\underline{V}\}$ be a formal series. We may write:

$$Q(\underline{V}) = \sum_{\underline{\lambda} \in (M^*)^n} c_{\underline{\lambda}} M(\underline{V})^{\underline{\lambda}}, \tag{12}$$

with $c_{\underline{\lambda}} \in \mathbb{C}$. Let $\Sigma : K \to \mathbb{R}^2$ be the embedding of $K$ given by $\nu \mapsto (\nu, \nu') \in \mathbb{R}^2$, let

$$\Sigma^n : K^n \to \mathbb{R}^{2n}$$

be the direct sum of $n$ copies of $\Sigma$.

The following Definition extends a Definition in section 3.4 of [Pel1].

**Definition 2.2** The *$K$-support* $\Sigma_K(Q)$ of an element $Q \in \mathbb{C}\{\underline{V}\}$ (12) is the subset of $(\Sigma(M^*))^n \subset \mathbb{R}^{2n}$ whose elements

$$\Sigma^n(\underline{\lambda}) = (\Sigma(\lambda_1), \dots, \Sigma(\lambda_n))$$

are such that $c_{\underline{\lambda}} \neq 0$. By definition, we set $\Sigma_K(0) = \emptyset$.

We will also use the $\mathbb{R}$-linear map

$$\begin{array}{rcl} \Pi : \mathbb{R}^{2n} & \to & \mathbb{R}^n \\ (y_1, y_1', \dots, y_n, y_n') & \mapsto & (y_1', \dots, y_n') \end{array}$$

(cf. p. 37 of [Pel1] in the case $n = 1$), so that if $\underline{\nu} = (\nu_1, \ldots, \nu_n)$ is in $K^n$,

$$\Pi(\Sigma^n(\underline{\nu})) = (\nu'_1, \ldots, \nu'_n) \in \mathbb{R}^n.$$

The following Lemma is an extension of Lemma 5.1 of [Pel1].

**Lemma 2.2 (A criterion of irrationality.)** *Let $Q(\underline{V})$ be an element of $\mathbb{C}\{\underline{V}\}$ converging in a non-empty neighbourhood $\mathcal{O}$ of $\underline{0}$. Let us suppose that the set $\Pi(\Sigma_K(Q))$ is contained in a strictly convex cone $\nabla$ included in $\mathbb{R}_{>0} \times \mathbb{R}^{n-1}$ and that it contains a sequence of points $(x'_{1,s}, \ldots, x'_{n,s}) \in (K \setminus \{0\})^n$ such that:*

$$\lim_{s \to \infty} (x'_{1,s}, \ldots, x'_{n,s}) = \underline{0} \in \mathbb{R}^n. \tag{13}$$

*Then, over $\mathcal{O}$, the series $Q$ does not converge to a rational function in $\mathbb{C}(\underline{V})$.*

**Proof.** Cauchy's multiplication of an element of $\mathbb{C}\{\underline{V}\}$ by a Laurent polynomial is well defined; let us check this for monomials first, taking into account the formalism that we have introduced. If $Q$ is as in (12) and if $\underline{\mu} = (\mu_1, \ldots, \mu_n) \in (M^*)^n$, then

$$M(\underline{V})^{\underline{\mu}} Q = \sum_{\underline{\lambda} \in (M^*)^n} c_{\underline{\lambda}} M(\underline{V})^{\underline{\lambda} + \underline{\mu}},$$

from which we also deduce that

$$\Sigma_K(M(\underline{V})^{\underline{\mu}} Q) = \Sigma^n(\underline{\mu}) + \Sigma_K(Q).$$

Let now $B$ be a non-zero Laurent polynomial of $\mathbb{C}\{\underline{V}\}$. We may write:

$$B(\underline{V}) \quad = \quad \sum_{\underline{\mu} \in \mathcal{E}} c_{\underline{\mu}} M(\underline{V})^{\underline{\mu}}, \tag{14}$$

where $\mathcal{E}$ is a finite non-empty subset in $(M^*)^n$ with $\Sigma^n(\mathcal{E}) = \Sigma_K(B)$, so that the $c_{\underline{\lambda}}$'s are all non-zero complex numbers. The Cauchy product of $B$ and $Q$ is the finite linear combination:

$$BQ(\underline{V}) = \sum_{\underline{\mu} \in \mathcal{E}} c_{\underline{\mu}} M(\underline{V})^{\underline{\mu}} Q(\underline{V})$$

(this product never vanishes).

The computation of $\Sigma_K(BQ)$ in terms of $\Sigma_K(Q)$ for a given polynomial $B$ is difficult (except if this polynomial is a monomial). At least, we can say that $\Sigma_K(BQ)$ is contained in the set

$$Z := \bigcup_{\underline{\lambda}\in\mathcal{E}} \Sigma_K(M(\underline{V})^{\underline{\lambda}}Q) \subset \mathbb{R}^{2n}. \tag{15}$$

The Lemma is proved if we show that, under our hypotheses, for any polynomial function $B(\underline{U}) \in \mathbb{C}[\underline{U}]$, the series $BQ$ does not converge to a polynomial function on $\mathcal{O}$, and this is equivalent to the fact that the $K$-support $\Sigma_K(BQ)$ of the series $BQ \in \mathbb{C}\{\underline{V}\}$ is an infinite set for every polynomial $B \in \mathbb{C}\{\underline{V}\}$.

By (13), the set $\Pi(\Sigma_K(Q))$ has $\underline{0}$ as an euclidean accumulation point. Moreover by hypothesis, for all $(y_1, \ldots, y_n) \in \Pi(\Sigma_K(Q))$, $y_1 > 0$. Therefore:

$$\inf\{y_1 \text{ such that } (y_1, y_2, \ldots, y_n) \in \Pi(\Sigma_K(Q))\} = 0.$$

Hence, for $\underline{\mu} = (\mu_1, \ldots, \mu_n) \in (M^*)^n$,

$$\inf\{y_1 \text{ such that } (y_1, y_2, \ldots, y_n) \in \Pi(\Sigma_K(M(\underline{V})^{\underline{\mu}}Q))\} = \mu'_1. \tag{16}$$

Let $B$ be any non-zero polynomial of $\mathbb{C}[\underline{V}]$ written as in (14); we have the set $Z$ defined as in (15), which contains $\Sigma_K(BQ)$, as said earlier.

The finiteness of $\mathcal{E}$ implies that there exists:

$$\iota := \inf\{y_1 \text{ such that } (y_1, y_2, \ldots, y_n) \in \Pi(Z)\}.$$

By (16), we see that

$$\iota = \min\{\mu'_1, \text{ such that } (\mu_1, \ldots, \mu_n) \in \mathcal{E}\}.$$

Let us denote by $\mathcal{E}_0$ the subset of $\mathcal{E}$ whose elements $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ are such that

$$\underline{\mu}' = (\mu'_1, \ldots, \mu'_n) = (\iota, *, \ldots, *) \in \mathbb{R}^n.$$

Let us choose an element $\underline{\mu}_0 = (\mu_{0,1}, \ldots, \mu_{0,n}) \in \mathcal{E}_0$. We now show that $\Sigma_K(BQ)$ contains infinitely many elements of $\Sigma^n(\underline{\mu}_0) + \Sigma_K(Q)$, and this will end the proof of the Lemma.

There exists $\epsilon > 0$ such that for all $\underline{\mu} \in \mathcal{E} \setminus \{\underline{\mu}_0\}$, the distance between $\underline{\mu}'_0$ and $\underline{\mu}' + \nabla$ is $> \epsilon$. This easily follows from Lemma 2.1 (point 2 if $\underline{\mu} \in \mathcal{E} \setminus \mathcal{E}_0$ or point 3 if $\underline{\mu} \in \mathcal{E}_0$) and the finiteness of $\mathcal{E}$. Hence,

$$B(\underline{\mu}'_0, \epsilon) \cap (\underline{\mu}' + \nabla) = \emptyset, \quad \text{for all } \underline{\mu} \in \mathcal{E} \setminus \{\underline{\mu}_0\} \tag{17}$$

(that is, the ball of $\mathbb{R}^n$ of center $\underline{\mu}_0'$ and radius $\epsilon$ has no common points with $\underline{\mu}' + \nabla$).

From (13), the ball $B(\underline{0}, \epsilon)$ contains infinitely many elements of $\Pi(\Sigma_K(Q))$: we denote by $\mathcal{Q}$ the infinite set of these elements.

The translate $\underline{\mu}_0' + \mathcal{Q}$ is also infinite, and

$$\underline{\mu}_0' + \mathcal{Q} \subset \underline{\mu}_0' + \Pi(\Sigma_K(Q)).$$

On the other hand,

$$(\underline{\mu}_0' + \mathcal{Q}) \quad \cap \quad (\underline{\mu}' + \Pi(\Sigma_K(Q))) = \emptyset, \text{ for all } \underline{\mu} \in \mathcal{E} \setminus \{\underline{\mu}_0\}$$

by (17). This implies:

$$\underline{\mu}_0 + \mathcal{Q} \subset \Pi(\Sigma_K(BQ)),$$

and the $K$-support of $BQ$ is infinite. □

## Multiplicative independence and vanishing Theorem.

**Definition 2.3** We say that the elements $\underline{z}_1, \ldots, \underline{z}_n \in \mathbb{C}^2$ are *$K$-linearly independent* if, writing $\underline{z}_i = (z_i, z_i')$ for all $i$, the only solution $(\mu_1, \ldots, \mu_n) \in K^n$ of the two simultaneous equations

$$\sum_{i=1}^{n} \mu_i \underline{z}_i = 0 \tag{18}$$

in $\mathbb{C}^2$ is the trivial solution $\mu_1 = \cdots = \mu_n = 0$. When $\underline{z}_1, \ldots, \underline{z}_n \in \mathbb{C}^2$ are $K$-linearly independent, we say that they form a *$K$-basis* of the $K$-vector space they generate. If $\underline{z}_1, \ldots, \underline{z}_n \in \mathbb{C}^2$ are not $K$-linearly independent, we will say that they are *$K$-linearly dependent*.

**Definition 2.4** We say that the points $\underline{a}_1, \ldots, \underline{a}_n \in \mathbb{T}$ are *multiplicatively independent* if the only solution $(\mu_1, \ldots, \mu_n) \in S^n$ of

$$\underline{a}_1^{\mu_1} \cdots \underline{a}_n^{\mu_n} = \underline{1} \tag{19}$$

in $\mathbb{T}$ is the trivial solution $\mu_1 = \cdots = \mu_n = 0$. A notion of multiplicative dependence is determined as well.

**Basic facts about $S$-modules and multiplicative independence.**

Before going on, we need a Remark.

**Remark 2.1** After the beginning of the proof of Lemma 2.1 of [Pel1], if $\alpha$ is an element of $K$, there always exists $n \in \mathbb{N} \setminus \{0\}$ such that $n\alpha \in S$.

**Lemma 2.3** *Let $\underline{a}_1, \ldots, \underline{a}_n$ be points of $\mathbb{T}$, let $\underline{z}_1 = (z_1, z'_1), \ldots, \underline{z}_n = (z_n, z'_n)$ be elements of $\mathbb{C}^2$ such that $\Phi(\underline{z}_i) = \underline{a}_i$ for $i = 1, \ldots, n$. The following three conditions are equivalent.*

1. *The points $\underline{a}_1, \ldots, \underline{a}_n$ are multiplicatively dependent.*
2. *The elements $\underline{1}, \underline{z}_1, \ldots, \underline{z}_n$ are $K$-linearly dependent.*
3. *There exists an $n$-tuple $\underline{\tau} = (\tau_1, \ldots, \tau_n) \in K^n \setminus \{\underline{0}\}$ and two elements $\nu_1, \nu_2 \in K$, linearly independent over $\mathbb{Q}$, such that:*

$$\sum_{i=1}^{n} (\nu_j \tau_i z_i + \nu'_j \tau'_i z'_i) \in \mathbb{Q} \quad \text{for } j = 1, 2. \tag{20}$$

**Proof.** The equivalence between the first two conditions of the lemma is easily checked because for all $\tau \in K$ there exists $p \in \mathbb{Z} \setminus \{0\}$ such that $p\tau \in S$ (Remark 2.1).

We prove that the third condition implies the second condition. Let $\nu_1, \nu_2$ be two elements of $K$ which are $\mathbb{Q}$-linearly independent and $\tau_1, \ldots, \tau_n \in K$, not all zero such that (20) holds. Hence, we have:

$$\sum_{i=1}^{n} (\nu_j \tau_i z_i + \nu'_j \tau'_i z'_i) = r_j, \quad j = 1, 2,$$

with $r_1, r_2 \in \mathbb{Q}$. It is enough to prove that, with $\underline{\tau} = (\tau_1, \ldots, \tau_n)$:

$$(\underline{\tau} \cdot {}^{\mathrm{t}}\underline{z}, \underline{\tau}' \cdot {}^{\mathrm{t}}\underline{z}') = \sum_{i=1}^{n} (\tau_i z_i, \tau'_i z'_i) \in \Sigma(K)$$

(the notation is the standard one for the multiplication of a line matrix by the transposed of a line matrix). Let us write $z_i = x_i + \mathrm{i} y_i$ with $x_i, y_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Let us write $a = \underline{\tau} \cdot {}^{\mathrm{t}}\underline{x}$, $a' = \underline{\tau}' \cdot {}^{\mathrm{t}}\underline{x}'$, $b = \underline{\tau} \cdot {}^{\mathrm{t}}\underline{y}$ and

$b' = \underline{\tau}' \cdot {}^{\mathrm{t}}\underline{y}'$ (where $\underline{x} := (x_1, \ldots, x_n)$, $\underline{y} := (y_1, \ldots, y_n)$ and similar meanings are attributed to $\underline{x}', \underline{y}'$). As the matrix $\begin{pmatrix} \nu_1 & \nu_1' \\ \nu_2 & \nu_2' \end{pmatrix}$ is invertible, from

$$\begin{pmatrix} \nu_1 & \nu_1' \\ \nu_2 & \nu_2' \end{pmatrix} \cdot \begin{pmatrix} a + \mathrm{i}b \\ a' + \mathrm{i}b' \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

separating real and imaginary part, we see that $b = b' = 0$ and:

$$\begin{aligned} \begin{pmatrix} a \\ a' \end{pmatrix} &= (\nu_1\nu_2' - \nu_2\nu_1')^{-1} \begin{pmatrix} \nu_2' & -\nu_1' \\ -\nu_2 & \nu_1 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ &= \Sigma\left(\frac{\nu_2' r_1 - \nu_1' r_2}{\nu_1\nu_2' - \nu_2\nu_1'}\right) \in \Sigma(K). \end{aligned}$$

This is the second condition, because we have showed that there exists $\tau \in K$ such that, in $\mathbb{C}$:

$$\sum_{i=1}^{n} \tau_i z_i = \tau \quad \text{and} \quad \sum_{i=1}^{n} \tau_i' z_i' = \tau'. \tag{21}$$

If on the other hand the relations (21) hold, then, we have $(\underline{\tau} \cdot {}^{\mathrm{t}}\underline{z}, \underline{\tau}' \cdot {}^{\mathrm{t}}\underline{z}') \in \Sigma(K)$, and for any two elements $\nu_1, \nu_2$ of $K$ we have (20). $\square$

**Algebraic interpretation of the property $A$.**

Let

$$\mathcal{C} = (a_{i,j})_{1 \leq i,j \leq \tilde{n}}$$

be a regular square matrix of order $\tilde{n}$ with its entries $a_{i,j}$ in $\mathbb{Z}$, let $\underline{V} = (v_1, \ldots, v_{\tilde{n}})$ be an element of $\mathbb{G}_{\mathrm{m}}^{\tilde{n}}$. We write

$$\mathcal{C}.\underline{V} = \underline{\tilde{V}} \in \mathbb{G}_{\mathrm{m}}^{\tilde{n}},$$

where $\underline{\tilde{V}} = (\tilde{v}_1, \ldots, \tilde{v}_{\tilde{n}})$ with $\tilde{v}_i = \prod_{j=1}^{\tilde{n}} v_j^{a_{i,j}}$.

Suppose that for all $1 \leq i, j \leq \tilde{n}$ we have $a_{i,j} \geq 0$. It is well known (p. 396 of [Lo-Po2]) that the maximum $\lambda_{\mathcal{C}}$ of the absolute values of the eigenvalues of $\mathcal{C}$ is itself an eigenvalue of $\mathcal{C}$.

**Definition 2.5** We say that the matrix $\mathcal{C}$ is *good* if it is non-singular, it has no roots of unit as eigenvalues, its minimal polynomial has all the eigenvalues of absolute value $= \lambda_{\mathcal{C}}$ as simple zeros, and it has an eigenvector $\underline{v}_{\mathcal{C}} \in \mathbb{R}^{\tilde{n}} \subset \mathbb{C}^{\tilde{n}}$ corresponding to $\lambda_{\mathcal{C}}$ whose coordinates are all positive (compare with [Mas2] p. 209).

If $\mathcal{C}$ is good, there exists a non-empty subset $\mathcal{U}(\mathcal{C})$ of $\mathbb{G}_{\mathrm{m}}^{\tilde{n}}$ such that for all $\underline{U} \in \mathcal{U}(\mathcal{C})$ we have

$$\lim_{k\to\infty} \mathcal{C}^k.\underline{U} = \underline{0}$$

in $\mathbb{C}^{\tilde{n}}$. The set $\mathcal{U}(\mathcal{C})$ is the intersection of an open euclidean neighbourhood of $\underline{0}$ in $\mathbb{C}^{\tilde{n}}$ with $\mathbb{G}_{\mathrm{m}}^{\tilde{n}}(\mathbb{C})$ (see definition 2 p. 93 of [Lo-Po1], or definition 2 and lemma 1 p. 397 of [Lo-Po2]). A good matrix $\mathcal{C}$ is one which satisfies the conditions I, II, III p. 33 of [Ni].

Let $\eta$ be an irrational unit of $S$ such that $\eta > 1 > \eta' > 0$. Let us consider the matrix:

$$\mathcal{C}(\eta) := \mathcal{B}(\eta) \oplus \cdots \oplus \mathcal{B}(\eta) \in \mathbf{SL}_{2n}(\mathbb{Z}),$$

which is the block diagonal matrix with copies of $\mathcal{B}(\eta)$ on its diagonal. It is easy to check that $\mathcal{C}(\eta)$ has its entries in $\mathbb{N}$, and that it is good. It is also easy to check that $\mathcal{D}^n \cap \mathbb{T}^n \subset \mathcal{U}(\mathcal{C}(\eta))$.

**Definition 2.6** If $\underline{A} \in \mathcal{U}(\mathcal{C}(\eta))$, we say that $\underline{A}$ *satisfies the property A for* $\mathcal{C}(\eta)$ ([1]) if the only locally analytic function $F$ such that $F(\mathcal{C}(\eta)^k.\underline{A}) = 0$ for all $k$ big enough is the zero function (p. 398 of [Lo-Po2]).

**Lemma 2.4** *Let us consider multiplicatively independent algebraic points* $\underline{a}_1, \dots, \underline{a}_n \in \mathbb{T} \cap \mathcal{D}$*. Then, the point*

$$\underline{A} := (\underline{a}_1, \dots, \underline{a}_n) \in \mathbb{T}^n \cap \mathcal{U}(\mathcal{C}(\eta))$$

*satisfies the property A for* $\mathcal{C}(\eta)$*.*

**Proof.** Assuming the hypotheses of the Lemma, let us suppose that $\underline{A}$ does not satisfy the property A. Theorem p. 276 of [Mas1] implies that there exist two distinct elements

$$(p_{1,1}, q_{1,1}, \dots, p_{n,1}, q_{n,1}), (p_{1,2}, q_{1,2}, \dots, p_{n,2}, q_{n,2}) \in \mathbb{N}^{2n},$$

and an arithmetic progression $\mathfrak{K} \subset \mathbb{N}$, such that if we write:

$$M_i(U_1, V_1, \dots, U_n, V_n) = U_1^{p_{1,i}} V_1^{q_{1,i}} \cdots U_n^{p_{n,i}} V_n^{q_{n,i}} \text{ for } i = 1, 2\ ,$$

we have:

$$M_1(\mathcal{C}(\eta)^\kappa.(u_1, v_1, \dots, u_n, v_n)) = M_2(\mathcal{C}(\eta)^\kappa.(u_1, v_1, \dots, u_n, v_n)) \quad \text{for all } \kappa \in \mathfrak{K}. \tag{22}$$

[1]This expression comes from the work of Loxton and van der Poorten. It probably abridges “analytic property”.

Let $\underline{z}_1, \ldots, \underline{z}_n \in \mathbb{C}^2$ be such that $\Phi(\underline{z}_i) = \underline{a}_i$ $(i = 1, \ldots, n)$, let us write $\underline{\mu}_i = \ell(p_{1,i}, q_{1,i}, \ldots, p_{n,i}, q_{n,i}) \in (M^*)^n$ $(i = 1, 2)$ so that

$$\Sigma^n(\underline{\mu}_i) = (p_{1,i}, q_{1,i}, \ldots, p_{n,i}, q_{n,i}) \cdot \mathfrak{B}^{\oplus n} \text{ for } i = 1, 2.$$

We observe that $\underline{\tau} = \underline{\mu}_1 - \underline{\mu}_2 \neq \underline{0}$. We have [2]:

$$\begin{aligned}
&M_i(\mathcal{C}(\eta)^\kappa.(u_1, v_1, \ldots, u_n, v_n)) = \\
&\quad = \quad e((p_{1,i}, q_{1,i}, \ldots, p_{n,i}, q_{n,i}) \cdot \mathcal{C}(\eta)^\kappa \cdot \mathfrak{B}^{\oplus n} \cdot {}^{\mathrm{t}}(z_1, z'_1, \ldots, z_n, z'_n)) \\
&\quad = \quad e((p_{1,i}, q_{1,i}, \ldots, p_{n,i}, q_{n,i}) \cdot \mathfrak{B}^{\oplus n} \cdot {}^{\mathrm{t}}(\eta^\kappa z_1, \eta^{-\kappa} z'_1, \ldots, \eta^\kappa z_n, \eta^{-\kappa} z'_n)) \\
&\quad = \quad e(\mathbf{t}(\eta^\kappa \underline{\mu}_i \cdot {}^{\mathrm{t}}\underline{z})) \text{ for } i = 1, 2.
\end{aligned}$$

Hence, the identities (22) for $\kappa \in \mathfrak{K}$ imply:

$$\mathbf{t}(\eta^{\kappa_1} \underline{\tau} \cdot {}^{\mathrm{t}}\underline{z}), \mathbf{t}(\eta^{\kappa_2} \underline{\tau} \cdot {}^{\mathrm{t}}\underline{z}) \in \mathbb{Q}$$

for two distinct rational integers $\kappa_1$ and $\kappa_2$. But since $\kappa_1, \kappa_2$ are distinct, $\eta^{\kappa_1}$ and $\eta^{\kappa_2}$ are $\mathbb{Q}$-linearly independent and Lemma 2.3 implies that $\underline{a}_1, \ldots, \underline{a}_n$ are multiplicatively dependent: a contradiction. □

**Remark 2.2** Lemma 2.4 can be turned into an equivalence, that is, we also have that if $\underline{a}_1, \ldots, \underline{a}_n$ are multiplicatively dependent, then the point $\underline{A}$ does not satisfy the property $A$ for $\mathcal{C}(\eta)$.

We give a short sketch of proof of this fact (we do not need it in the proof of our Theorem). If $\underline{a}_1, \ldots, \underline{a}_n \in \mathcal{D}$ are multiplicatively dependent, then it is possible to find $l \in \mathbb{N}^\times$ such that the point

$$\underline{A}' = \underline{A}^l = (\underline{a}_1^l, \ldots, \underline{a}_n^l)$$

lies in a connected algebraic subgroup $H$ of $\mathbb{T}^n = \mathbb{G}_{\mathrm{m}}^{2n}$ of even codimension $\geq 2$, which also is an $S$-module with the diagonal action of $S$ (over $\mathbb{T}^n$). In particular, for all $k \in \mathbb{Z}$,

$$\underline{A}'^{\eta^k} \in H.$$

Let $H_1$ be a hypersurface containing $H$; for all $k$:

$$\mathcal{C}(\eta)^k.\underline{A}' = \underline{A}'^{\eta^k} \in H_1.$$

We can choose $H_1$ so that an equation defining it also provides a non zero locally analytic function $F$ such that for all $k$ big enough,

$$F(\mathcal{C}(\eta)^k.\underline{A}') = 0.$$

This implies that the point $\underline{A}$ does not satisfy the property $A$ for $\mathcal{C}(\eta)$.

[2] By using the notation $\mathbf{t}(\alpha\underline{z}) = \alpha z + \alpha' z'$, for $\alpha \in K$ and $\underline{z} = (z, z')$.

**Positive bases for $S$-modules of finite rank.**

**Lemma 2.5** *Let $\beta$ be an element of $S$. The map $\underline{u} \mapsto \underline{u}^{\beta}$ from $\mathbb{T}$ to $\mathbb{T}$ extends to an analytic map $\mathbb{C}^2 \to \mathbb{C}^2$ if and only if*

$$\beta \geq \max\left\{\frac{\theta'}{\theta}\beta', \beta'\right\}. \tag{23}$$

**Proof.** Let $\nu$ be an element of $M^*$. By using the notation (11), The function $\underline{u} \mapsto M(\underline{u})^{\nu}$ from $\mathbb{T}$ to $\mathbb{G}_{\mathrm{m}}$ extends to an analytic function $\mathbb{C}^2 \to \mathbb{C}$ if and only if the exponents of the monomial above are positive, if and only if

$$\nu \geq \max\left\{-\nu', -\frac{\theta}{\theta'}\nu'\right\} \tag{24}$$

(this follows from [Pel1, Lemma 3.1, condition 2]).

From the definition of the exponential function $\Phi$ we see that

$$\underline{u}^{\beta} = (M(\underline{u})^{\delta^{-1}\beta}, M(\underline{u})^{\delta^{-1}\theta'^{-1}\beta}).$$

Hence, the map $\underline{u} \mapsto \underline{u}^{\beta}$ extends to an analytic map $\mathbb{C}^2 \to \mathbb{C}^2$ if and only if $\nu = \delta^{-1}\beta, \delta^{-1}\theta'^{-1}\beta$ satisfy (24).

The first condition is equivalent to $\delta^{-1}\beta \geq \max\{\delta^{-1}\beta', -(\theta/\theta')\delta^{-1}\beta'\}$, that is, $\beta \geq \max\{\beta', -(\theta/\theta')\beta'\}$. The second condition is equivalent to $-\delta^{-1}\theta'^{-1}\beta \geq \max\{-\delta^{-1}\theta^{-1}\beta', -\delta^{-1}\theta'^{-1}\beta'\}$, that is, $\beta \geq \max\{(\theta'/\theta)\beta', \beta'\}$.

Therefore, both conditions simultaneously hold if and only if

$$\beta \geq \max\left\{\beta', -\frac{\theta}{\theta'}\beta', \frac{\theta'}{\theta}\beta'\right\}.$$

But after the condition (1), $0 < -(\theta/\theta') < 1$ and $\theta'/\theta < -1$. The proof of the Lemma is complete by considering two cases, following the sign of $\beta'$, in the inequality above. $\square$

Let $\mathcal{P}$ be the set of the elements $\beta \in K$ such that (23) holds. We notice that $\mathcal{P} \supset \mathbb{Q}$ and, more generally, if $\beta \in K$ is such that $\beta \geq |\beta'| > 0$, then $\beta \in \mathcal{P}$. Moreover, if $\beta \in K$ is such that $\beta > 0$, then for all $\eta \in S$ unit big enough with $\eta > 1 > \eta' > 0$, we have $\eta\beta \in \mathcal{P}$.

We will need the following Proposition, which is an extension to the "real multiplication" of [Ni, Lemma 3.4.9 p. 105] (see also [Lo-Po3, Lemma 3 p. 37]).

**Proposition 1** *Let $\underline{u}_1, \ldots, \underline{u}_m$ be elements of $\mathbb{T} \cap \mathcal{D}^+$. It is possible to find an $n$-tuple (with $1 \leq n \leq m$)*

$$\underline{A} = (\underline{a}_1, \ldots, a_n) \in (\mathbb{T} \cap \mathcal{D}^+)^n$$

*with $\underline{a}_1, \ldots \underline{a}_n$ multiplicatively independent, such that for some torsion points $\underline{g}_1, \ldots, \underline{g}_m \in \mathbb{T}$, units $\eta_1, \ldots, \eta_m$ of $S$ with $\eta_i > 1 > \eta_i' > 0$ and elements $\nu_{i,j} \in S \cap \mathcal{P}$ $(i = 1, \ldots, m, j = 1, \ldots, n)$ we have:*

$$\underline{u}_i^{\eta_i} = \underline{g}_i \underline{a}_1^{\nu_{i,1}} \cdots \underline{a}_n^{\nu_{i,n}}, \quad (i = 1, \ldots, m). \tag{25}$$

The proof of this Proposition requires the following Lemma.

**Lemma 2.6** *Let $\underline{w}_1, \ldots, \underline{w}_m$ be elements of $\mathcal{W}^+$. There exist: an integer $n$ such that $m \geq n \geq 1$, elements $\alpha_1, \ldots, , \alpha_m \in K$, elements $\underline{t}_i \in \mathcal{W}^+$ $(i = 1, \ldots, n)$ with $\underline{1}, \underline{t}_1, \ldots, \underline{t}_n$ $K$-linearly independent, and elements $\mu_{i,j} \in S$ such that*

$$\underline{w}_i = \alpha_i \underline{1} + \sum_{j=1}^{n} \mu_{i,j} \underline{t}_j, \quad (i = 1, \ldots, m). \tag{26}$$

*Moreover, there also exist: units $\eta_1, \ldots, \eta_m \in W$ with $\eta_i > 1$ for all $i$, elements $\underline{z}_i \in \mathcal{W}^+$ with $\underline{1}$ and $\underline{z}_1, \ldots, \underline{z}_n$ $K$-linearly independent and elements $\nu_{i,j} \in S \cap \mathcal{P}$ $(1 \leq i \leq m, 1 \leq j \leq n)$, such that*

$$\eta_i \underline{w}_i = \eta_i \alpha_i \underline{1} + \sum_{j=1}^{n} \nu_{i,j} \underline{z}_j, \quad (i = 1, \ldots, m). \tag{27}$$

**Proof.** We remark that the set $\Sigma(K)$ is dense in $\mathbb{R}^2$ so that for any couple $\underline{x} \in \mathbb{R}^2$ it is possible to find $\nu \in K$ with $(\nu, \nu')$ close enough to $\underline{x}$.

We also point out that if $(x, x') \in \mathcal{W}^+$ and if $(0, 0) \neq (y, y') \in \overline{\mathcal{W}^+}$ (the euclidean closure of $\mathcal{W}^+$), then $(x + y, x' + y'), (xy, x'y') \in \mathcal{W}^+$. Moreover, for all $\alpha \in \mathcal{P}$, if $\underline{s} \in \mathcal{W}^+$, then $\alpha \underline{s} \in \mathcal{W}^+$. This holds in particular for $\alpha = \eta$ a unit such that $\eta \geq 1$ and for $\alpha = \Delta$ a positive rational number.

The first part of the Lemma is easy. We consider the $K$-vector space generated by $\underline{1}, \underline{w}_1, \ldots, \underline{w}_m$ (with the action of $K$ that we have considered until now). Since $\underline{w}_i \notin \mathbb{R}^2$ for all $i$, its dimension $n + 1$ satisfies $2 \leq n + 1 \leq m + 1$. We can find a $K$-basis $\underline{1}, \underline{\widetilde{t}}_1, \ldots, \underline{\widetilde{t}}_n$ with $\underline{\widetilde{t}}_j \in \mathcal{W}^+$, and we have relations

$$\underline{w}_i = \alpha_i \underline{1} + \sum_{j=1}^{n} \tau_{i,j} \underline{\widetilde{t}}_j, \quad (i = 1, \ldots, m)$$

with $\alpha_i, \tau_{i,j} \in K$ for all $i, j$.

By Remark 2.1, there exists $\Delta \in \mathbb{N} \setminus \{0\}$ such that $\mu_{i,j} = \Delta\tau_{i,j} \in S$ for all $i, j$. By setting $\underline{t}_j = \Delta^{-1}\underline{\widetilde{t}}_j$ for all $j$, we get relations such as (26), because $\underline{t}_j \in \mathcal{W}^+$ for all $j$.

We now prove the second part of the Lemma. We prove the existence of relations such as (27) by induction on $m$, since the property is evident for $m = 1$ where it suffices to take $\underline{z}_1 = \underline{w}_1$, since $1 \in S \cap \mathcal{P}$.

We may assume without loss of generality that (26) holds for $i = 1, \dots, m-1$ with $\mu_{i,j} \in S \cap \mathcal{P}$ for all $j$ (to simplify the notations, we write $\underline{w}_i$ at the place of the $\eta_i\underline{w}_i$'s obtained by the induction hypothesis). By (26), we also have:

$$\underline{w}_m = \alpha_m\underline{1} + \sum_{j=1}^{n} \mu_{m,j}\underline{t}_j, \tag{28}$$

but the elements $\mu_{m,j}$ of $S$ do not necessarily belong to $\mathcal{P}$.

However, if we reorder the elements $\underline{t}_i$ in a suitable way, we can assume that there exist two integer $r, s$ such that $1 \leq r \leq s \leq n$, and such that

$$\underline{w}_m = \alpha_m\underline{1} + \sum_{j=1}^{r} \mu_{m,j}\underline{t}_j + \sum_{j=s+1}^{n} \mu_{m,j}\underline{t}_j,$$

with $\mu_{m,j} > 0$ for $j = 1, \dots, r$ and $\mu_{m,j} < 0$ for $j = s+1, \dots, n$ (sums with empty sets of terms vanish by convention).

Let us also choose elements $\rho_{r+1}, \dots, \rho_m \in K$ such that $\rho_j > 0$ and such that

$$\underline{\widetilde{w}}_m := \underline{w}_m - \sum_{j=r+1}^{m} \rho_j\underline{t}_j \in \mathcal{W}^+.$$

This is certainly possible and we have infinitely many choices; it suffices to choose $\rho_j$ small enough for all $j$. Of course, we also have $-\mu_{m,j} + \rho_j > 0$ $(j = s+1, \dots, n)$.

Let $\eta_1$ be a unit of $S$ such that $\eta_1 > 1 > \eta_1' > 0$, let us write:

$$\begin{aligned}
\lambda_j &= \eta_1\mu_{m,j}, \quad (1 \leq j \leq r), \\
&= \eta_1\rho_j, \quad (r+1 \leq j \leq s), \\
&= -\eta_1\mu_{m,j} + \eta_1\rho_j, \quad (s+1 \leq j \leq n),
\end{aligned}$$

so that for all $j = 1, \dots, n$, we have $\lambda_j > 0$.

If $\eta_1$ is big enough, then $0 < \eta_1' < 1$ is very small and we can suppose that

$$\lambda_j \in \mathcal{P}, \quad (j = 1, \dots, n).$$

Moreover, we have:

$$\eta_1 \underline{\widetilde{w}}_m = \eta_1 \alpha_m \underline{1} + \sum_{i=1}^{r} \lambda_i \underline{t}_i - \sum_{j=r+1}^{n} \lambda_j \underline{t}_j. \tag{29}$$

We claim that there exist elements $\sigma_{i,j} \in K$ $(1 \le i \le r, r+1 \le j \le n)$ with $\sigma_{i,j} > 0$ such that

$$\sum_{i=1}^{r} \sigma_{i,j} = \lambda_j, \quad (j = r+1, \ldots, n)$$

and

$$\Im\left(\lambda_i t_i - \sum_{j=r+1}^{n} \sigma_{i,j} t_j\right) > 0, \quad (i = 1, \ldots, r), \tag{30}$$

where $\underline{t}_i = (t_i, t_i')$ for all $i$.

Let $G := \sum_{i=1}^{r} \lambda_i \Im(t_i)$; it is a positive non-zero real number. If $\sigma_{i,j} \in K$ are non-negative and very close to

$$\frac{\lambda_i \Im(t_i) \lambda_j}{G}$$

for $i = 1, \ldots, r-1$ and for all $j$, then, again for all $j$, $\sigma_{r,j} := \lambda_j - \sum_{i=1}^{r-1} \sigma_{i,j}$ is very close to $\lambda_r \Im(t_r) \lambda_j / G$, non-negative and in $K$. The claim can be easily checked with these choices.

We take the $\sigma_{i,j}$'s as in the claim and $\eta_2$ a unit of $S$ very big (with $\eta_2 > 1 > \eta_2' > 0$), so that $0 < \eta_2' < 1$ is very small. Let $\Delta$ be a positive rational integer, let us define $\underline{z}_1, \ldots, \underline{z}_n \in \mathbb{C}^2$ (depending on $\eta_2$ and $\Delta$) by the relations:

$$\begin{aligned} \Delta \underline{z}_i &= \eta_2 \underline{t}_i - \sum_{j=r+1}^{n} \frac{\eta_2 \sigma_{i,j}}{\lambda_i} \underline{t}_j \quad (i = 1, \ldots, r), \\ \Delta \underline{z}_j &= \underline{t}_j, \quad (j = r+1, \ldots, n), \end{aligned}$$

that is:

$$\eta_2 \underline{t}_i = \Delta \underline{z}_i + \sum_{j=r+1}^{n} \frac{\eta_2 \sigma_{i,j} \Delta}{\lambda_i} \underline{z}_j \quad (i = 1, \ldots, r), \tag{31}$$

$$\underline{t}_j = \Delta \underline{z}_j, \quad (j = r+1, \ldots, n). \tag{32}$$

Thanks to (30), we can choose $\eta_2 \in W$ big enough and a positive integer $\Delta$ so that $\underline{z}_i \in \mathcal{W}^+$ $(i = 1, \dots, r)$ and the following conditions hold simultaneously:

$$\frac{\eta_2 \Delta \sigma_{i,j}}{\lambda_i}, \Delta\lambda_i, \eta_2 \Delta \rho_j \in S \cap \mathcal{P}, \quad (i = 1, \dots, r, j = r+1, \dots, n). \tag{33}$$

We also have $\underline{z}_j \in \mathcal{W}^+$ for $j = r+1, \dots, n$ (this follows directly from the fact that $\underline{t}_i \in \mathcal{W}^+$ for all $i$). Hence, $\underline{z}_i \in \mathcal{W}^+$ for $i = 1, \dots, n$ and $\underline{1}, \underline{z}_1, \dots, \underline{z}_n$ are $K$-linearly independent because the $\underline{t}_i$'s satisfy this property already.

Let $\eta$ be equal to $\eta_1\eta_2$. From (29), (31) and (32), we see that

$$\begin{aligned} \eta \underline{\widetilde{w}}_m &= \eta \alpha_m \underline{1} + \sum_{i=1}^{r} \lambda_i (\eta_2 \underline{t}_i) - \sum_{j=r+1}^{n} \sum_{i=1}^{r} (\eta_2 \sigma_{i,j}) \underline{t}_j \\ &= \eta \alpha_m \underline{1} + \sum_{i=1}^{r} \Delta \lambda_i \underline{z}_i. \end{aligned}$$

From the definition of $\underline{\widetilde{w}}_m$ we get

$$\eta \underline{\widetilde{w}}_m = \eta \left( \underline{w}_m - \sum_{j=r+1}^{n} \rho_j \Delta \underline{z}_j \right),$$

and we see that

$$\eta \underline{w}_m = \eta \alpha_m \underline{1} + \sum_{i=1}^{r} \Delta \lambda_i \underline{z}_i + \sum_{j=r+1}^{n} \eta \rho_j \Delta \underline{z}_j,$$

which is, after (33), a linear combination of the $\underline{z}_j$'s with coefficients $\nu_{m,j} \in S \cap \mathcal{P}$.

At the same time, after (31), (32) and the conditions (33), the elements $\eta_2 \underline{t}_1, \dots, \eta_2 \underline{t}_r, \underline{t}_{r+1}, \dots, \underline{t}_n$ are linear combinations of the $\underline{z}_j$'s with coefficients $\varsigma_{i,l} \in (S \cap \mathcal{P}) \cup \{0\}$. Hence, for $k = 1, \dots, m-1$, we have:

$$\begin{aligned} \eta_2 \underline{w}_k &= \eta_2 \alpha_k \underline{1} + \sum_{i=1}^{r} \mu_{k,i} (\eta_2 \underline{t}_i) + \sum_{j=r+1}^{n} (\eta_2 \mu_{k,j}) \underline{t}_j \\ &= \eta_2 \alpha_k \underline{1} + \sum_{l=1}^{n} \varsigma_{i,l} \left( \sum_{i=1}^{r} \mu_{k,i} \Delta + \sum_{j=r+1}^{n} \eta_2 \mu_{k,j} \right) \underline{z}_l. \end{aligned}$$

Since the $\mu_{k,j}$'s are in $S \cap \mathcal{P}$ by induction hypothesis, since the $\varsigma_{i,l}$'s, if nonzero, satisfy the same property, and since by (31) and (32) one sees that for each $i$ there exists $l$ with $\varsigma_{i,l} \neq 0$, the relations above are also linear combinations of the $\underline{z}_j$'s with non-zero coefficients $\nu_{i,k}$, hence in $S \cap \mathcal{P}$. $\square$

**Proof of Proposition 1.** We recall that $\mathcal{D}^+ \cap \mathbb{T} = \Phi(\mathcal{W}^+)$. Let us choose elements $\underline{w}_i = (w_i, w_i') \in \mathcal{W}^+$ $(i = 1, \ldots, m)$ such that $\Phi(\underline{w}_i) = \underline{u}_i$ for all $i$. By using Lemma 2.6, we get units $\eta_1, \ldots, \eta_m$, elements $\alpha_1, \ldots, \alpha_m \in K$, elements $\underline{z}_j$ $(j = 1, \ldots, n$ for a certain $n$ with $1 \leq n \leq m)$ satisfying certain properties.

If we set $\underline{g}_i = \Phi(\alpha_i \eta_i \underline{1})$ for all $i$ and $\underline{a}_j = \Phi(\underline{z}_j)$ for all $j$, then we get the relations (25). $\square$

# 3 Proof of our Theorem.

A straightforward computation shows that our Theorem 1 is equivalent to the following Theorem, that we will prove in this section.

**Theorem 4** *Let $\underline{u}_1, \ldots, \underline{u}_m$ be algebraic elements of $\mathbb{T} \cap \mathcal{D}$, let $\{\mathcal{I}_1, \ldots, \mathcal{I}_r\}$ be the partition of $\{1, \ldots, m\}$ associated to the $S$-equivalence. If the numbers*

$$f(\underline{u}_1), \ldots, f(\underline{u}_m)$$

*are algebraically dependent, there exists $s$ with $1 \leq s \leq r$ such that the numbers $\{f(\underline{u}_i), i \in \mathcal{I}_s\}$ are algebraically dependent.*

The plan of the proof is the following. We suppose that the numbers $f(\underline{u}_1), \ldots, f(\underline{u}_m)$ are algebraically dependent. We first check that a criterion of algebraic independence of [Ni] applies to a collection of functions $\Psi_1, \ldots, \Psi_m$ we are going to define below. The field generated by $\mathbb{Q}$ and the images of these functions at a certain point $\underline{A}$ of $\mathbb{T}^n$ has the same algebraic closure as the field generated by $\mathbb{Q}$ and $f(\underline{u}_1), \ldots, f(\underline{u}_m)$. After this criterion of algebraic independence (which needs some tools introduced in section 2 to apply), the functions $\Psi_1, \ldots, \Psi_m$ themselves are algebraically dependent (over rational functions).

In section 3 we use the tools of section 2 again to prove that necessarily, for some $s$ such that $1 \leq s \leq r$, the functions $\Psi_i$ with $i \in \mathcal{I}_s$ are linearly dependent "modulo rational functions" and we will reach the conclusion.

For all $\underline{u} \in \mathcal{D}$ there exists a constant $c > 1$ such that if $\eta > c$ is a unit of $S$ with $1 > \eta' > 0$, then $\underline{u}^\eta \in \mathcal{D}^+$ (in fact, $c$ can be chosen independently on $\underline{u}$ but we do not need this here). Hence, after the functional equation (5), there is no loss of generality to choose $\underline{u}_1, \ldots, \underline{u}_m$ in $\mathcal{D}^+$. By the way, the reader can check that the action of units has no influence on the $S$-equivalence classes.

We now apply Proposition 1. We have a positive integer $n$, multiplicatively independent algebraic points $\underline{a}_1, \ldots, \underline{a}_n \in \mathbb{T}\cap\mathcal{D}^+$, elements $\nu_{i,j} \in S\cap\mathcal{P}$ $(1 \leq i \leq m, 1 \leq j \leq n)$, torsion points $\underline{g}_1, \ldots, \underline{g}_m$ such that the equalities (25) hold.

We may arrange the indexes $i = 1, \ldots, m$ so that there exists $0 \leq m_0 \leq m$ with:

$$\begin{array}{lcll} \nu'_{i,1} & > & 0 & \text{for } i = 1, \ldots, m_0, \\ \nu'_{i,1} & < & 0 & \text{for } i = m_0+1, \ldots, m. \end{array} \tag{34}$$

According with the value of $m_0$, we define:

$$\begin{aligned} \Psi_i(\underline{V}) = \Psi_i(\underline{v}_1, \ldots, \underline{v}_n) &= f^+(\underline{g}_i \underline{v}_1^{\nu_{i,1}} \cdots \underline{v}_n^{\nu_{i,n}}) \text{ for } i = 1, \ldots, m_0, \\ &= f(\underline{g}_i \underline{v}_1^{\nu_{i,1}} \cdots \underline{v}_n^{\nu_{i,n}}) \text{ for } i = m_0+1, \ldots, m, \end{aligned} \tag{35}$$

where $f^+$ is the twin series of $f$ defined in section 3.1 of [Pel1].

**Proposition 2** *The functions $\Psi_1, \ldots, \Psi_m$ are algebraically dependent on $\mathbb{C}(\underline{V})$.*

**Proof.** By the point 2 of Proposition 3.5 of [Pel1], the sum of $f$ and $f^+$ is a rational function $\Theta$ analytic on $\mathcal{D}^+$ and defined over $\mathbb{Q}$. Hence we have the equality of subfields of $\mathbb{C}$:

$$\begin{aligned} &\mathbb{Q}(\underline{u}_1, \ldots, \underline{u}_m, f(\underline{u}_1), \ldots, f(\underline{u}_m)) = \\ &\quad = \mathbb{Q}(\underline{u}_1, \ldots, \underline{u}_m, f(\underline{u}_1), \ldots, f(\underline{u}_{m_0}), f^+(\underline{u}_{m_0+1}), \ldots, f^+(\underline{u}_m)). \end{aligned}$$

By the functional equation (5) and by the functional equation of $f^+$ (see equations (3.9), (3.10) of [Pel1]), we check the equality of subfields of $\mathbb{C}$:

$$\overline{\mathbb{Q}}(f(\underline{u}_1), \ldots, f(\underline{u}_m)) = \overline{\mathbb{Q}}(\Psi_1(\underline{A}), \ldots, \Psi_m(\underline{A})),$$

where $\overline{\mathbb{Q}}$ denotes an embedding of an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

The Proposition follows from [Ni, Theorem 3.3.2] (or a criterion of algebraic independence of Loxton and van der Poorten in [Lo-Po2]), if we check some properties of the functions $\Psi_i$; we do this now.

First of all, since $\nu_{i,j} \in S \cap \mathcal{P}$, the maps

$$(\underline{v}_1, \ldots, \underline{v}_n) \mapsto \underline{g}_i \underline{v}_1^{\nu_{i,1}} \cdots \underline{v}_n^{\nu_{i,n}}$$

extend to analytic functions $\mathbb{C}^{2n} \to \mathbb{C}^2$ by Lemma 2.5.

Since for $\underline{v}_1, \ldots, \underline{v}_n \in \mathcal{D}^+ \cap \mathbb{T}$ we also have

$$\underline{g}_i \underline{v}_1^{\nu_{i,1}} \cdots \underline{v}_n^{\nu_{i,n}} \in \mathcal{D}^+ \cap \mathbb{T},$$

and since $f, f^+$ simultaneously converge on $\mathcal{D}^+$, the series $\Psi_1, \ldots, \Psi_m$ converge on $\mathcal{D}^{+n}$, hence on $\underline{A} = (\underline{a}_1, \ldots, \underline{a}_n)$, which belongs to $\mathcal{D}^{+n}$.

The Taylor expansions at $\underline{0}$ of the functions $\Psi_1, \ldots, \Psi_n$ have their coefficients in the cyclotomic number field generated by $\mathbb{Q}$ and the coordinates of the torsion points $\underline{g}_1, \ldots, \underline{g}_m$, because the functions $f, f^+$ have their Taylor expansion at $\underline{0}$ defined over $\mathbb{Q}$.

By Lemma 4.6 of [Pel1], there exists an infinite subgroup $W$ of the multiplicative group of totally positive units of $S$ such that for all $\eta \in W$ we have $\underline{g}_i^\eta = \underline{g}_i$ $(i = 1, \ldots, m)$. After the functional equations (3.9) and (3.10) of [Pel1], the functions $\Psi_i(\underline{V})$ satisfy on $\mathcal{D}^{+n}$ the collection of simultaneous functional equations:

$$\Psi_i(\mathcal{C}(\eta).\underline{V}) = \Psi_i(\underline{V}) - R_i(\underline{V}), \tag{36}$$

for $\eta \in W$ with $\eta > 1 > \eta' > 0$, where

$$\begin{aligned} R_i(\underline{V}) &= R_\eta^+(\underline{g}_i \underline{v}_1^{\nu_{i,1}} \cdots \underline{v}_n^{\nu_{i,n}}) \text{ for } i = 1, \ldots, m_0, \\ &= R_\eta(\underline{g}_i \underline{v}_1^{\nu_{i,1}} \cdots \underline{v}_n^{\nu_{i,n}}) \text{ for } i = m_0 + 1, \ldots, m. \end{aligned}$$

The series $R_\eta^+, R_\eta$ are defined in section 3.5.2 of [Pel1], where it is proved that they converge on $\mathcal{D}^+$ to rational functions defined over $\mathbb{Q}$. Hence the series $R_i$ are all convergent on $\mathcal{D}^{+n}$ to rational functions defined over a cyclotomic number field of finite dimension.

The matrix $\mathcal{C}(\eta)$ involved in the automorphism of (36) is good, as we said earlier, at the beginning of section 2, thanks to the fact that $\eta > 1$. Moreover, by Lemma 2.4, the point $\underline{A}$ satisfies the property $A$, (these conditions altogether mean, as we said earlier, that the conditions I, II, III, IV on p. 33-34 of [Ni] are satisfied). The mentioned Proposition of [Ni] applies. $\square$

## Conclusion of the proof.

We recall a notation from [Pel1]. If $E$ is a subset of $K$, then

$$\begin{aligned} E_+ &= \{\nu \in E \text{ such that } \nu > 0, \nu' > 0\}, \\ E_\pm &= \{\nu \in E \text{ such that } \nu > 0, \nu' < 0\}. \end{aligned}$$

Theorem 3.2.2 of [Ni] (or Corollary 9 p. 29 of [Ku]) implies that the functions $\Psi_i(\underline{V})$ are $\mathbb{C}$-*linearly dependent modulo* $\mathbb{C}(\underline{V})$. In other words, there exist $m$ complex numbers $c_1, \ldots, c_m$, not all zero, and a rational function $Q^+(\underline{V}) \in \mathbb{C}(\underline{V})$ such that:

$$\sum_{i=1}^{m} c_i \Psi_i(\underline{V}) = Q^+(\underline{V}). \tag{37}$$

Before going further, let us collect some information about the functions $\Psi_i$ and $Q^+$.

**Lemma 3.1** *For all $i = 1, \ldots, m$ we have*

$$\Sigma_K(\Psi_i) \quad \subset \quad \Sigma^n(\underline{\nu}_i M^*) \tag{38}$$

*and there exists a strictly convex cone $\nabla \subset \mathbb{R}_{>0} \times \mathbb{R}^{n-1}$ such that*

$$\Pi(\Sigma_K(\Psi_1)), \ldots, \Pi(\Sigma_K(\Psi_n)), \Pi(\Sigma_K(Q^+)) \quad \subset \quad \nabla. \tag{39}$$

**Proof.** From (34), the description of the $K$-supports of $f$ and $f^+$ given by formulas (3.12), (3.14) of [Pel1] (take $N = M$) and the definitions (35), we see that

$$\begin{aligned} \Sigma_K(\Psi_i) \quad &= \quad \{\Sigma^n(\nu_{i,1}\mu, \ldots, \nu_{i,n}\mu) \text{ for all } \mu \in M^* \text{ such that } \mu > \mu' > 0\} \\ &\qquad (i = 1, \ldots, m_0), \qquad (40) \\ \Sigma_K(\Psi_i) \quad &= \quad \{\Sigma^n(\nu_{i,1}\mu, \ldots, \nu_{i,n}\mu) \text{ for all } \mu \in M^* \text{ such that } \mu > -\mu' > 0\} \\ &\qquad (i = m_0 + 1, \ldots, m). \qquad (41) \end{aligned}$$

Since $\underline{\nu}_i \in S^n$ for all $i$, we clearly have (38).

If $i = 1, \ldots, m_0$ then $\nu_{i,1} \in S_+$ and we have $\nu_{i,1}\mu \in M_+^*$ for all $\mu \in M_+^*$. If $i = m_0+1, \ldots, m$ then $\nu_{i,1} \in S_\pm$ and $\nu_{i,1}\mu \in M_+^*$ for all $\mu \in M_\pm^*$. In all cases, by (40) and (41), we see that if $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is such that $\Sigma^n(\underline{\lambda}) \in \Sigma_K(\Psi_i)$ for some $i$, then $\lambda_1' > 0$. Hence, for each $i$, the set $\Pi(\Sigma_K(\Psi_i))$ is contained in the half-lines:

$$\begin{aligned} (\nu_{i,1}', \ldots, \nu_{i,n}')\mathbb{R}_{>0} \quad &\subset \quad \mathbb{R}_{>0} \times \mathbb{R}^{n-1} \quad (i = 1, \ldots, m_0), \\ (\nu_{i,1}', \ldots, \nu_{i,n}')\mathbb{R}_{<0} \quad &\subset \quad \mathbb{R}_{>0} \times \mathbb{R}^{n-1} \quad (i = m_0 + 1, \ldots, m). \end{aligned}$$

Thus, there exists a strictly convex cone $\nabla \subset \mathbb{R}_{>0} \times \mathbb{R}^{n-1}$ (of axis $\underline{Y} = (1, 0, \ldots, 0)\mathbb{R}_{>0}$) such that for all $i = 1, \ldots, m$

$$\Pi(\Sigma_K(\Psi_i)) \subset \nabla$$

and for every choice of $(c_1, \ldots, c_m) \in \mathbb{C}^m \setminus \{\underline{0}\}$, we also have $\Pi(\Sigma_K(Q^+)) \subset \nabla$, and we get (39). $\square$

**Lemma 3.2** *If $\Sigma_K(\Psi_i) \cap \Sigma_K(\Psi_j) \neq \emptyset$, then $\underline{\nu}_i = (\nu_{i,1}, \ldots, \nu_{i,n})$, $\underline{\nu}_j = (\nu_{j,1}, \ldots, \nu_{j,n})$ are $K$-linearly dependent and $\underline{u}_i \sim \underline{u}_j$ ($\underline{u}_i$, $\underline{u}_j$ are $S$-equivalent).*

**Proof.** The first property follows from (38) because for all $j$, the elements $\underline{\nu}$ with $\Sigma^n(\underline{\nu}) \in \Sigma_K(\Psi_j)$ are multiples of $\underline{\nu}_j$ with proportionality factor in $M^* \subset K$. After (25) we also see that $\Gamma_{\text{sat}}(\underline{u}_i) = \Gamma_{\text{sat}}(\underline{u}_j)$ (equality of saturated $S$-modules; Definition 1.1), that is, $\underline{u}_i \sim \underline{u}_j$. $\square$

Thanks to Lemma 3.2, there exist pairwise $K$-linearly independent elements $\underline{\tau}_1, \ldots, \underline{\tau}_r \in (K \setminus \{0\})^n$ with $\underline{\tau}_s = (\tau_{s,1}, \ldots, \tau_{s,n})$ and $\tau'_{s,1} > 0$ for $1 \leq s \leq r$ such that, $\{\mathcal{I}_1, \ldots, \mathcal{I}_r\}$ being the partition of $\{1, \ldots, m\}$ associated to the $S$-equivalence, if $i \in \mathcal{I}_s$ then $\underline{\nu}_i = \varsigma_i \underline{\tau}_s$ where $\varsigma_i \in (K \setminus \{0\})$ is such that $\varsigma'_i > 0$ if $\nu'_{i,1} > 0$, (that is, if $i = 1, \ldots, m_0$) and $\varsigma'_i < 0$ if $\nu'_{i,1} < 0$ (that is, if $i = m_0 + 1, \ldots, m$).

After Remark 2.1, We can choose two positive rational integers $q', q''$ such that, for all $i = 1, \ldots, m$ and $s = 1, \ldots, r$:

$$\begin{aligned} \beta_i := q' \varsigma_i &\in S \setminus \{0\}, \\ \underline{\gamma}_s := q'' \underline{\tau}_s &\in (S \setminus \{0\})^n. \end{aligned}$$

Let $q = q'q''$. We have:

$$q \underline{\nu}_i = (q' \varsigma_i)(q'' \underline{\tau}_s) = \beta_i \underline{\gamma}_s, \tag{42}$$

with $i \in \mathcal{I}_s$.

We easily check that if $1 \leq i \leq m_0$ then $\beta_i \in S_+$, and if $m_0 + 1 \leq i \leq m$ then $\beta_i \in S_\pm$. Let us write:

$$\begin{aligned} \phi_i^+(\underline{u}) &= f^+(\underline{g}_i \underline{u}^{\beta_i}), \quad \text{if } i = 1, \ldots, m_0, \\ \phi_i^+(\underline{u}) &= f(\underline{g}_i \underline{u}^{\beta_i}), \quad \text{if } i = m_0 + 1, \ldots, m. \end{aligned}$$

Using (42) we have, for $1 \leq i \leq m_0$ and $i \in \mathcal{I}_s$:

$$\begin{aligned} \Psi_i(\underline{v}_1^q, \ldots, \underline{v}_n^q) &= f^+(\underline{g}_i \underline{v}_1^{q\nu_{i,1}} \cdots \underline{v}_n^{q\nu_{i,n}}) \\ &= f^+(\underline{g}_i (\underline{v}_1^{\gamma_{s,1}} \cdots \underline{v}_n^{\gamma_{s,n}})^{\beta_i}) \\ &= \phi_i^+(\underline{v}_1^{\gamma_{s,1}} \cdots \underline{v}_n^{\gamma_{s,n}}), \end{aligned}$$

and we have a similar equality for $m_0+1 \le i \le m$. Thus, for all $i = 1, \dots, m$:

$$\Psi_i(\underline{v}_1^q, \dots, \underline{v}_n^q) = \phi_i^+(\underline{v}_1^{\gamma_{s,1}} \cdots \underline{v}_n^{\gamma_{s,n}}), \tag{43}$$

where $s$ is the integer such that $i \in \mathcal{I}_s$.

By Proposition 3.5 points 3, 5, 6 of [Pel1], the series $\phi_i^+$ satisfy

$$\Sigma_K(\phi_i^+) \subset N_i^* \cap K_+, \quad i = 1, \dots, m$$

where $N_i := \beta_i^{-1} M$.

We now need the following Lemma, which is the key result of this section.

**Lemma 3.3** *Let us assume that (37) holds for non-zero complex numbers $c_1, \dots, c_m$ and $Q^+(\underline{V})$ a rational function, let $(\mathcal{I}_s)_{s=1,\dots,r}$ be the partition of $\{1, \dots, m\}$ associated to the $S$-equivalence. Then, for all $s$:*

$$\sum_{i \in \mathcal{I}_s} c_i \phi_i^+(\underline{u}) = Q_s^+(\underline{u}), \tag{44}$$

*where $Q_s^+(\underline{u})$ is a rational function.*

**Proof.** There is no loss of generality to suppose that $c_i \neq 0$ for $i = 1, \dots, m$.

Since $\beta_i > 0$ for all $i$, there exists a unit $\eta \in S_+$ with $\eta > 1$ such that $\eta\beta_i \ge |\eta'\beta_i'| > 0$ for all $i$. By Proposition 3.5 point 6, the series $\phi_i^+(\underline{u}^\eta)$ converge on $\mathcal{D}^+$ (remember that $\mathcal{D}^+ \subset \mathcal{D}$).

Since for all $s$, condition (44) is equivalent to

$$\sum_{i \in \mathcal{I}_s} c_i \phi_i^+(\underline{u}^\eta) = Q_s^+(\underline{u}^\eta)$$

and $Q_s^+(\underline{u}^\eta)$ is rational if and only if $Q_s^+(\underline{u})$ is rational, we can obviously suppose, in (44), that the series $\phi_i^+$ are convergent over $\mathcal{D}^+$. Obviously, $Q_s^+(\underline{u})$ (being a linear combination of series having appropriate properties) converge on $\mathcal{D}^+$ too.

We argue by contradiction. Let us suppose that there exists $1 \le s \le r$ such that the series $Q_s^+(\underline{u})$ of (44) does not converge to a rational function.

Comparing (44) with equation (5.4) of [Pel1], we can apply Proposition 5.2 of this reference. From the point (2) of this Proposition we see that there exists a sequence of non-zero elements:

$$(x_i')_{i \in \mathbb{N}} \subset \Pi(\Sigma_K(Q_s^+))$$

such that $\lim_{i \to \infty} x_i' = 0$.

The $K$-supports of the series

$$Q_s^+(\underline{v}_1^{\gamma_{s,1}}\cdots\underline{v}_n^{\gamma_{s,n}})$$

are pairwise disjoint for $s=1,\ldots,r$ (Lemma 3.2). Combining (43) and (37) we see, with $\underline{V}^q=(\underline{v}_1^q,\ldots,\underline{v}_n^q)$, that

$$\sum_{i=1}^m c_i\phi_i^+(\underline{v}_1^{\gamma_{s(i),1}}\cdots\underline{v}_n^{\gamma_{s(i),n}})=\sum_{s=1}^r Q_s^+(\underline{v}_1^{\gamma_{s,1}}\cdots\underline{v}_n^{\gamma_{s,n}})=Q^+(\underline{V}^q),$$

where $s(i)$ is the integer such that $i\in\mathcal{I}_{s(i)}$. There exists a sequence of points:

$$((x'_{1,l},\ldots,x'_{n,l}))_{l\in\mathbb{N}}\subset\Pi(\Sigma_K(Q^+(\underline{V}^q)))$$

such that (13) holds. Thanks to (39), Lemma 2.2 applies (we know already that $Q^+(\underline{V})$ converge on a non-empty open neighbourhood of $\underline{0}\in\mathbb{C}^{2n}$), and $Q^+(\underline{V}^q)$ converges to an irrational function. This implies that $Q^+(\underline{V})$ converges to an irrational function: a contradiction. $\square$

Combining Lemma 3.3 and [Pel1, Proposition 5.2 point 3] we see that our assumptions imply the existence of a non-trivial relation in $\mathbb{C}\{\underline{U}\}$:

$$\sum_{i\in\mathcal{I}_s} c_i f_{N_i}^+(\Phi((\alpha_i/\beta_i)\underline{1})\underline{U})=0,$$

where $\Phi(\alpha_i\underline{1})=\underline{g}_i$, $N_i=\beta_i^{-1}M$ and $f_{N_i}^+$ is defined in section 3.5.1 of [Pel1] (for all $i$).

By [Pel1, Proposition 4.4], we can assume that $c_1,\ldots,c_m\in\mathbb{Q}$ and $Q_s^+(\underline{V})\in\overline{\mathbb{Q}}(\underline{V})$ in (44) for all $s$. We now choose $s$ such that for some $i\in\mathcal{I}_s$, $c_i\neq 0$. Without loss of generality, we assume that $s=1$. We have (in the notations of [Pel1, Proposition 4.4]):

$$\sum_{i\in\mathcal{I}_1} l_i\mathbb{F}(\Phi(\alpha_i\underline{1})\underline{U}^{\beta_i})=0,$$

for rational numbers not all zero $l_i$.

Let us consider the point:

$$\underline{v}=\Phi\left(\sum_{j=1}^n \tau_{1,j}\underline{z}_j\right),$$

where $\Phi(\underline{z}_i) = \underline{a}_i$ for all $i$. We see that:

$$\begin{aligned} \underline{v}^{q''} &= \Phi\left(\sum_{j=1}^{n} q''\tau_{1,j}\underline{z}_j\right) \\ &= \underline{a}_1^{\gamma_{1,1}} \cdots \underline{a}_n^{\gamma_{1,n}}, \end{aligned}$$

so that $\underline{v}$ is defined over $\overline{\mathbb{Q}}$. Let $i \in \mathcal{I}_1$:

$$\begin{aligned} \underline{g}_i \underline{v}^{q''\beta_i} &= \underline{g}_i \underline{a}_1^{\nu_{i,1}} \cdots \underline{a}_n^{\nu_{i,n}} \\ &= \underline{u}_i^{\eta_i}, \end{aligned}$$

by (42) and (25). By Proposition 4.8 of [Pel1], there exist rational numbers $d_i$ $(i \in \mathcal{I}_1)$ and an algebraic number $h$ with

$$\sum_{i \in \mathcal{I}_1} d_i f(\underline{u}_i^{\eta_i}) = h.$$

Applying the functional equation (5) we end the proof of our Theorem. $\square$

# 4 Proof of the Corollaries.

In this section we prove the Corollaries 1 and 2, then we relate our results to earlier works in Mahler's method.

### Proof of Corollary 1; preliminary Lemmata.

We prove Corollary 1 in the case $w = \theta$ with $\theta$ satisfying (1). The reader can obtain the general case applying the arguments of the introduction, or those on pp. 3-4 of [Pel1].

The hypothesis on $H$ is equivalent to the existence of a couple of coprime rational integers $(h, l) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that for all $(u, v) \in H$, $u^l v^h = 1$.

Let us write $\nu = \delta^{-1}(-h\theta'^{-1} + l) \in M^*$; we have that for all $r \in \mathbb{Q} \setminus \mathbb{Z}$, $r\nu \notin M^*$. Moreover, $\underline{u} = \Phi(z, z') \in H$ (for complex numbers $z, z'$) if and only if

$$\nu z + \nu' z' \in \Sigma(M) \subset \mathbb{C}^2. \tag{45}$$

Thanks to our Theorem 4 (equivalent to the main Theorem), we can suppose that for all $i, j$ such that $1 \leq i, j \leq m$, $\underline{u}_i \sim \underline{u}_j$, or in other words, that the the $m$-tuple $\mathcal{M} = (\underline{u}_1, \ldots, \underline{u}_m)$ satisfies the rank one hypothesis (section 2.3 of [Pel1]).

Hence, there exists a non-torsion algebraic point $\underline{a} \in \mathbb{T}$ such that:

$$\underline{u}_i = \underline{k}_i \underline{a}^{\beta_i}, \quad i = 1, \ldots, m, \tag{46}$$

for torsion points $\underline{k}_i \in \mathbb{T}$ and $\beta_i \in S \setminus \{0\}$. We must prove that there exists $1 \leq i \neq j \leq m$ such that $\underline{u}_i = \underline{u}_j$.

**Lemma 4.1** *With the hypotheses above (in particular if (46) holds for some non-torsion algebraic point $\underline{a} \in \mathbb{T}$), there exists a non torsion algebraic point $\underline{w} \in H$, torsion points $\underline{g}_i \in H$ and non-vanishing positive rational integers $r_i$, such that for all $i = 1, \ldots, m$:*

$$\underline{u}_i = \underline{g}_i \underline{w}^{r_i}, \quad i = 1, \ldots, m. \tag{47}$$

**Proof.** We first remark that the Lemma follows if we prove it without the condition on the positivity of the $r_i$'s. Indeed, these numbers, if they exist, must be non-zero and of the same sign, as $\mathcal{D}$ has the property that if $\underline{u} \in \mathcal{D}$, then $\underline{g}\underline{u}^{-1} \notin \mathcal{D}$ for all torsion points $\underline{g}$, and torsion points lie on its boundary. If they all are negative, we can replace $\underline{w}$ with $\underline{w}^{-1}$.

We claim that if two non-torsion algebraic elements $\underline{u}, \underline{v}$ of $H \cap \mathcal{D}$ satisfy

$$\underline{u}^{\beta} \underline{v}^{\gamma} = \underline{1}, \tag{48}$$

for some $\beta, \gamma \in S \setminus \{0\}$, then there exists an algebraic non-torsion point $\underline{w} \in H$ such that:

$$\underline{u} = \underline{g}\underline{w}^{a}, \quad \underline{v} = \underline{w}^{b}, \tag{49}$$

with a torsion point $\underline{g} \in H$, and $a, b \in \mathbb{N} \setminus \{0\}$.

Let us consider $\underline{t}_1 = (t_1, t_1'), \underline{t}_2 = (t_2, t_2') \in \mathbb{C}^2$ such that:

$$\Phi(\underline{t}_1) = \underline{u}, \quad \Phi(\underline{t}_2) = \underline{v}.$$

Let $x_1, x_2, y_1, y_2, x_1', \ldots \in \mathbb{R}$ be such that

$$t_i = x_i + \mathrm{i} y_i, \quad t_i' = x_i' + \mathrm{i} y_i', \quad (i = 1, 2).$$

The relation (48) and (45) (insert $\underline{z} = \underline{t}_i$ in the latter) imply:

$$\begin{aligned} \beta y_1 + \gamma y_2 &= \beta' y_1' + \gamma' y_2' = 0, && (50) \\ \nu y_1 + \nu' y_1' &= \nu y_2 + \nu' y_2' = 0. && (51) \end{aligned}$$

We see that $y_1 y_1' y_2 y_2' \neq 0$: indeed, if for example, $y_1 = 0$, then by (51) we obtain $y_1' = 0$ and by some of the equalities (4.10) of [Pel1], $\underline{u}$ lies in the boundary of $\mathcal{D}$, case that has been excluded.

Now, from (50) we see that $\beta/\gamma = -y_2/y_1 \in K^\times$ and $\beta'/\gamma' = -y_2'/y_1'$. Moreover, from (51), $y_2/y_1 = y_2'/y_1'$, so that $\beta/\gamma = \beta'/\gamma'$, that is, $\beta/\gamma \in \mathbb{Q}^\times$. There exist non-vanishing rational integers $p, q$ such that:

$$\frac{\beta}{\gamma} = \frac{q}{p}.$$

From the relation $\underline{u}^{p\beta}\underline{v}^{p\gamma} = \underline{1}$ we get $\underline{u}^{q\gamma}\underline{v}^{p\gamma} = \underline{1}$ which implies $\underline{u}^q\underline{v}^p = \underline{b}$ for some torsion point $\underline{b} \in H$.

Let $\underline{w} \in H$ be an element such that $\underline{w}^q = \underline{v}$: it is a non-torsion algebraic point because $\underline{v}$ is algebraic, non-torsion. From $\underline{u}^q = \underline{b}\underline{w}^{-pq}$ we get $\underline{u} = \underline{g}\underline{w}^{-p}$ for some torsion point $\underline{g} \in H$ such that $\underline{g}^p = \underline{b}$; the claim is proved.

We end the proof of Lemma 4.1 by induction on $m > 0$. If $m = 1$ the property to be proved is trivially satisfied. Let us suppose that

$$\underline{u}_i = \underline{p}_i \underline{x}^{s_i}, \quad i = 1, \ldots, m-1,$$

where $\underline{p}_i$ are torsion points and $s_i$ are integers. We apply the claim (equality (49)), to $\underline{u} = \underline{u}_m$ and $\underline{v} = \underline{x}$. There exist a torsion point $\underline{g}$ of $H$ and a non-torsion algebraic point $\underline{w} \in H$ such that $\underline{u}_m = \underline{g}\underline{w}^a$ and, for all $i = 1, \ldots, m-1$:

$$\begin{aligned} \underline{u}_i &= \underline{g}\underline{p}_i \underline{w}^{a s_i} \\ &= \underline{g}_i \underline{w}^{r_i}, \end{aligned}$$

and we set $r_m = a$. □

For the next Lemma we introduce the following class of formal double Laurent series in $\mathbb{C}\{\underline{U}\}$ (see section 4 of [Pel1]). If $N$ is a free $\mathbb{Z}$-submodule of $K$ of rank 2 containing $M$ (so that $N^*$ is contained in $M^*$), we write:

$$\mathbb{F}_N(\underline{U}) = \sum_{\nu \in N^* \setminus \{0\}} M(\underline{U})^\nu.$$

If $\beta \in K^\times$ is such that $\beta N \subset N$ (that is, if $\beta \in S(N)$) and if $\underline{g}$ is a torsion point in $\mathbb{T}$, then the series

$$\mathbb{F}_N(\underline{g}\underline{U}^\beta) = \sum_{\nu \in \beta N^* \setminus \{0\}} M(\underline{g}^\sharp \underline{U})^\nu = \mathbb{F}_{\beta^{-1}N}(\underline{g}^\sharp \underline{U})$$

(for each $\underline{g}^\sharp$ torsion point such that $\underline{g}^{\sharp\beta} = \underline{g}$) in $\mathbb{C}\{\underline{U}\}$ is well defined. These equalities follow from the arguments in the beginning of the proof of Proposition 4.4 of [Pel1].

The next Lemma may be used to show that the dimension of the $\mathbb{Q}$-vector space spanned by the series in (1.5) of [Pel1] does not depend on the choice of $\underline{v}$ in (1.4) of loc. cit. This fact naturally follows from Theorem 1.1 of loc. cit. because one sees that the change of the generator $\underline{v}$ has not influence on the degree of transcendence of $\mathcal{K}$, and this degree is also the mentioned dimension (by the way, this independence on $\underline{v}$ can also be read on the saturated $S$-modules). In [Pel1] we wrote this as a remark without proof; since this property is needed in the proof of Corollary 1, after the statement we will also give a short (direct) proof.

**Lemma 4.2** *Let $(\underline{u}_1, \ldots, \underline{u}_m)$ be a $m$-tuple of $\mathbb{T}^m$ whose coefficients are non-torsion, let us suppose that there exists elements $\alpha_{i,j} \in K$, two non-torsion points $\underline{v}_1, \underline{v}_2 \in \mathbb{T}$ and elements $\beta_{i,j} \in S \setminus \{0\}$ ($i = 1, \ldots, m, j = 1, 2$) such that:*

$$\underline{u}_i = \Phi(\alpha_{i,1}\underline{1})\underline{v}_1^{\beta_{i,1}} = \Phi(\alpha_{i,2}\underline{1})\underline{v}_2^{\beta_{i,2}}, \quad i = 1, \ldots, m. \tag{52}$$

*Then we have (with $N_{i,j} = \beta_{i,j}^{-1}M$):*

$$\sum_i c_i \mathbb{F}_{N_{i,1}} \left( \Phi(\alpha_{i,1}\underline{1})\underline{U}^{\beta_{i,1}} \right) = 0 \tag{53}$$

*if and only if*

$$\sum_i c_i \mathbb{F}_{N_{i,1}} \left( \Phi(\alpha_{i,2}\underline{1})\underline{U}^{\beta_{i,2}} \right) = 0. \tag{54}$$

**Proof.** We only need to prove one implication. Since $\underline{v}_1, \underline{v}_2$ are non-torsion, we can choose elements $\underline{z}_1 = (z_1, z_1')$, $\underline{z}_2 = (z_2, z_2') \in \mathbb{C}^2 \setminus \Sigma(K)$ such that:

$$\underline{v}_1 = \Phi(\underline{z}_1), \quad \underline{v}_2 = \Phi(\underline{z}_2).$$

Moreover, there exists $\tilde{\beta}_1, \tilde{\beta}_2 \in S \setminus \{0\}$ such that

$$\underline{v}_1^{\tilde{\beta}_1} \underline{v}_2^{\tilde{\beta}_2} = \underline{1},$$

so that there exists $\delta \in K \setminus \{0\}$ and $\gamma \in K$ with

$$\underline{z}_1 = \delta \underline{z}_2 + \gamma \underline{1}.$$

The conditions (52) are equivalent to the expressions in $\mathbb{T}$ ($i = 1, \ldots, m$):

$$\underline{u}_i = \Phi(\alpha_{i,1}\underline{1} + \beta_{i,1}\underline{z}_1) = \Phi(\alpha_{i,2}\underline{1} + \beta_{i,2}\underline{z}_2).$$

These settings imply, for all $i$:

$$\alpha_{i,1} + \beta_{i,1}\gamma - \alpha_{i,2} = \tau_i \in M \text{ and } \beta_{i,1}\delta = \beta_{i,2}. \tag{55}$$

The relation (53) is equivalent to the following linear relation of formal Fourier series (see section 3.3 of [Pel1] and use Lemma 3.4 of this reference) for formal variables $\underline{Z}_1 = (Z_1, Z_1'), \underline{Z}_2 = (Z_2, Z_2')$ such that $\underline{Z}_1 = \delta\underline{Z}_2 + \gamma\underline{1}$:

$$\begin{aligned}
0 &= \sum_{i=1}^m c_i \sum_{\nu\in N_{i,1}^*} e\left(\mathbf{t}\left(\nu\left(\frac{\alpha_{i,1}}{\beta_{i,1}}\underline{1} + \underline{Z}_1\right)\right)\right) \\
&= \sum_{i=1}^m c_i \sum_{\nu\in N_{i,1}^*} e\left(\mathbf{t}\left(\nu\left(\frac{\alpha_{i,1}}{\beta_{i,1}}\underline{1} + \delta\underline{Z}_2 + \gamma\underline{1}\right)\right)\right) \\
&= \sum_{i=1}^m c_i \sum_{\nu\in N_{i,2}^*} e\left(\mathbf{t}\left(\nu\left(\frac{\alpha_{i,1}+\beta_{i,1}\gamma}{\beta_{i,1}\delta}\underline{1} + \underline{Z}_2\right)\right)\right) \\
&= \sum_{i=1}^m c_i \sum_{\nu\in N_{i,2}^*} e\left(\mathbf{t}\left(\nu\left(\frac{\alpha_{i,2}+\tau_i}{\beta_{i,2}}\underline{1} + \underline{Z}_2\right)\right)\right) \\
&= \sum_{i=1}^m c_i \sum_{\nu\in N_{i,2}^*} e\left(\mathbf{t}\left(\nu\left(\frac{\alpha_{i,2}}{\beta_{i,2}}\underline{1} + \underline{Z}_2\right)\right)\right),
\end{aligned}$$

thanks to (55) and because $\tau_i/\beta_{i,2} \in N_{i,2}$. Again by Lemma 3.4 of [Pel1], (54) holds. □

### End of proof of Corollary 1.

We apply Lemma 4.1 (to the identities (46)). Let $\underline{w}, r_1, \ldots, r_m, \underline{g}_1, \ldots, \underline{g}_m$ be as in (47).

Since $f(\underline{u}_1), \ldots, f(\underline{u}_m)$ are algebraically dependent, By Theorem 1.1 of [Pel1] there exist rational numbers $c_1, \ldots, c_m$ not all zero, such that (53) holds, with $\beta_{i,1} = \beta_i$, $\underline{k}_i = \Phi(\alpha_{i,1}\underline{1})$ for $i = 1, \ldots, m$. By setting $\beta_{i,2} = r_i$, $\underline{g}_i = \Phi(\alpha_{i,2}\underline{1})$ (for all $i$) and applying Lemma 4.2, we see that (54) holds.

This relation implies, as for the relation (4.4) of [Pel1], the linear relations for all $\mu \in M^*$:

$$\sum_{\substack{i \text{ such that} \\ \mu \in r_i M^*}} c_i e(\mathbf{t}(\alpha_i\mu)/r_i) = 0. \tag{56}$$

Let $\overline{N} \subset K$ be the $\mathbb{Z}$-module generated by all the elements $\alpha \in K$ such that $\mathbf{t}(\alpha\nu) \in \mathbb{Z}$ ($\nu$ being defined by (45)); then $\overline{N} = \mathbb{Q}\alpha^\sharp + M$, for some $\alpha^\sharp \in K \setminus \{0\}$ ($h, l$ are coprime).

Since $\underline{g}_i \in H$ for all $i$, we have $\alpha_1, \ldots, \alpha_m \in \overline{N}$. Let $N$ be the $\mathbb{Z}$-module generated by $M, \alpha_1, \ldots, \alpha_m$ so that $M \subset N \subset \overline{N}$. The group $N/M$ is finite and cyclic: let $\alpha \in N$ be a representative of a generator. There exists a positive rational integer $\ell$ such that $\ell\alpha \in M$ and $s\alpha \notin M$ for $s = 1, \ldots, \ell - 1$. There exists $m$ rational integers $0 \leq s_1, \ldots, s_m \leq \ell - 1$ such that $\alpha_i \in s_i\alpha + M$. For all $\mu \in M^*$, the relations (56) become:

$$\sum_{\substack{i \text{ such that} \\ \mu \in r_i M^*}} c_i e(\mathbf{t}(\alpha\mu)s_i/r_i) = 0.$$

We note that the image of the map $\Upsilon : M^* \to \mathbb{Q}$ defined by $\mu \mapsto \mathbf{t}(\alpha\mu)$ is $\mathbb{Z}/\ell$. Indeed, it is possible to construct a basis $(\omega, \tau)$ of $M^*$ such that the dual basis of $M$ is $(\kappa, \ell\alpha)$ for some $\kappa$.

Let us choose $r = r_{i_0}$ the smallest possible with $c_{i_0} \neq 0$. Then, $\Upsilon$ maps the set $rM^* \setminus \cup_{r_j \neq r} r_j M^*$ surjectively onto $(r/\ell)\mathbb{Z}$. The relations (56) imply:

$$\sum_{\substack{i \text{ such that} \\ r_i = r}} c_i e(s_i h/\ell) = 0, \text{ for all } h \in \mathbb{Z}.$$

The vanishing of a Vandermonde determinant implies, for two distinct indices $i, j$, that $r_i = r_j$ and $s_i = s_j$. This finally gives $\underline{u}_i = \underline{u}_j$, hence proving Corollary 1. □

### Proof of Corollary 2.

Let us consider an $m$-tuple $\mathcal{M} = (\underline{u}_1, \ldots, \underline{u}_m)$ of algebraic elements of $\mathbb{T} \cap \mathcal{D}$ such that the $S$-module they generate is free (not containing torsion-points other than $\underline{1}$). After Theorem 1, we can suppose that $\mathcal{M}$ satisfies the rank one hypothesis [Pel1, section 2.3]. There exists an algebraic non-torsion point $\underline{v} \in \mathbb{T}$ such that

$$\underline{u}_i = \underline{v}^{\beta_i}, \quad i = 1, \ldots, m$$

with the $\beta_i$'s in $S \setminus \{0\}$. By Theorem 1.1 of [Pel1], the formal double Laurent series

$$\mathbb{F}_{\beta_i M}(\underline{U})$$

are $\mathbb{Q}$-linearly dependent, and this means that for $1 \leq i \neq j \leq m$, $\beta_i M = \beta_j M$. Hence $\beta_i/\beta_j M = M$, which means that $\eta = \beta_i/\beta_j$ is a unit of $S$, hence proving Corollary 2. □

**Link with earlier results of Loxton and van der Poorten.**

In the introduction we quoted a very general result of Loxton and van der Poorten which also implies some properties of algebraic independence of values of Hecke-Mahler series $f(u,v)$ with algebraic $(u,v) \in \mathbb{T} \cap \mathcal{D}$. The interesting application of this result to the series $f$ appears on pp. 407-408 of [Lo-Po2]. Corollary 2 implies these properties and suggests future studies in Mahler's method.

**Definition 4.1** Let $(u_1, v_1), \ldots, (u_m, v_m)$ be couples of non-zero algebraic numbers. We will say that the $m$-tuple $((u_1, v_1), \ldots, (u_m, v_m))$ is *free* if the following is true. Let $\eta$ be a unit of $S$ such that $\eta > 1 > \eta' > 0$, let $(p_1, q_1, \ldots, p_m, q_m)$ be any $2m$-tuple of rational integers and let us define the sequences $(p_j^{(k)}, q_j^{(k)}) = (p_j, q_j) \cdot \mathcal{B}(\eta)^k$ for $j = 1, \ldots, m$ and $k \in \mathbb{N}$. If

$$\prod_{i=1}^m u_i^{p_i^{(k)}} \prod_{i=1}^m v_i^{q_i^{(k)}} = 1$$

for infinitely many $k \geq 0$, then $p_1 = q_1 = \cdots = p_m = q_m = 0$.

**Theorem 5 (Loxton and van der Poorten)** *Let* $((u_1, v_1), \ldots, (u_m, v_m))$ *be an $m$-tuple of couples of algebraic numbers such that* $0 < |u_j||v_j|^\theta < 1$ *and* $|u_j| < 1$ *for all* $j = 1, \ldots, m$. *If the $m$-tuple* $((|u_1|, |v_1|), \ldots, (|u_m|, |v_m|))$ *is free, the complex numbers* $f(u_1, v_1), \ldots, f(u_m, v_m)$ *are algebraically independent over* $\mathbb{Q}$.

This Theorem contains a well known Theorem of Mahler in [Mah] (case $m = 1$). By the proof of Lemma 2.4, we see that an $m$-tuple $(\underline{u}_1, \ldots, \underline{u}_m)$ is free if and only if $\underline{u}_1, \ldots, \underline{u}_m$ are multiplicatively independent. Now, an $m$-tuple $(\underline{u}_1, \ldots, \underline{u}_m)$ of multiplicatively independent algebraic elements of $\mathbb{T} \cap \mathcal{D}$ clearly satisfies the hypotheses of Corollary 2 which implies the algebraic independence of $f(\underline{u}_1), \ldots, f(\underline{u}_m)$.

Moreover, if an $m$-tuple $\mathcal{M} = ((u_1, v_1), \ldots, (u_m, v_m))$ of couples of algebraic numbers is such that the $m$-tuple $((|u_1|, |v_1|), \ldots, (|u_m|, |v_m|))$ is free, then $\mathcal{M}$ is also free: this shows that our Corollary 2 implies Theorem 5.

There exist plenty of $m$-tuples $(\underline{u}_1, \ldots, \underline{u}_m)$ with $f(\underline{u}_1), \ldots f(\underline{u}_m)$ algebraically independent, but not free (examples are given in the next section). Therefore, our results are much more precise than Theorem 5.

# 5 Appendix: a portrait of the relations.

We divide the relations in two different types.

**Relations of generic type.**

There are linear relations, that we call “generic”, which hold for all the $w \in \mathbb{R}_{>0}$ at once, regardless to their quadraticity and the algebraicity of the points. These relations, all homogeneous and defined over $\mathbb{Q}$, arise from the linear relations of rational functions:

$$\sum_{i=1}^{m} c_i f_\infty(g_i u^{b_i}, g_i v^{b_i}) = 0,$$

where $c_1, \ldots, c_m$ are rational numbers, $g_1, \ldots, g_m$ are roots of unity, $b_1, \ldots, b_m$ are positive rational integers, and $f_\infty = \lim_{w \mapsto \infty} f_w$ is the rational function

$$f_\infty(u, v) = \frac{uv}{(1-u)(1-v)}.$$

If $(u, v)$ is a couple of complex numbers in $\mathbb{T} \cap \mathcal{D}$, the simplest example is the relation (1.8) of [Pel1] with $m = 5$ (notice that the quadruple $((u, v), (u, -v), (-u, v), (-u, -v))$ is not free).

**Relations of special type.**

There are linear relations that we call “special”, which hold for irrational quadratic $w$’s. these relations are often non-homogeneous, defined over $\overline{\mathbb{Q}}$ but not necessarily on $\mathbb{Q}$. If $w = \theta$ satisfies (1), the simplest example of these relations is provided for $m = 2$ by the functional equation (5).

There are many other special relations. Let us consider an irrational element $\beta \in S$ such that $\beta \geq |\beta'| > 0$. Let $\mathrm{Ker}(\beta)$ be the kernel of the group homomorphism $\mathbb{T} \to \mathbb{T}$ defined by $\underline{u} \mapsto \underline{u}^\beta$: this kernel has $\mathbf{n}(\beta)$ elements, indeed there is an isomorphism of groups $\mathrm{Ker}(\beta) \cong (\beta^{-1}M/M)$. Choose any algebraic element $\underline{v} \in \mathbb{T} \cap \mathcal{D}$ and consider the $(\mathbf{n}(\beta)+1)$-tuple of elements of $\mathbb{T} \cap \mathcal{D}$:

$$(\underline{u}_0, \underline{u}_1 \ldots, \underline{u}_{\mathbf{n}(\beta)}) = (\underline{v}^\beta, \underline{g}_1\underline{v}, \ldots, \underline{g}_{\mathbf{n}(\beta)}\underline{v}),$$

where $\underline{g}_1, \ldots, \underline{g}_{\mathbf{n}(\beta)}$ are the elements of $\mathrm{Ker}(\beta)$. With the arguments of section 4.2 of [Pel1], it is easy to check the existence of a relation of special type:

$$\mathbf{n}(\beta) f(\underline{u}_0) - \sum_{i=1}^{m} f(\underline{u}_i) = l \in \overline{\mathbb{Q}}.$$

Note that $l$ might be non-zero, and it is easy to produce examples with $l \neq 0$ in this case.

Moreover, the $\mathbb{Q}$-linear independence of the formal double Laurent series

$$\mathbb{F}_M(\underline{g}_i \underline{U}), \quad i = 1, \ldots, \mathbf{n}(\beta)$$

can be easily checked following the arguments of section 7 of [Pel1], and by Theorem 1.1 of [Pel1] we get the algebraic independence of the numbers:

$$f(\underline{u}_1), \ldots, f(\underline{u}_{\mathbf{n}(\beta)}).$$

This result cannot be reached neither by Theorem 2 nor by Theorem 5 because in the first result one coordinate of the points must be 1 and in the second the $\mathbf{n}(\beta)$-tuple

$$(\underline{u}_1 \ldots, \underline{u}_{\mathbf{n}(\beta)}) = (\underline{g}_1 \underline{v}, \ldots, \underline{g}_{\mathbf{n}(\beta)} \underline{v})$$

must be free, which is false.

In this text we have studied the case of $w$ quadratic only, essentially because we needed the functional equation (5) to apply Mahler's method. This leads to many other relations of special type and our results imply that there are relations of generic and special type only. What happens for non-quadratic $w$'s?

For a non-quadratic $w > 0$, only relations of generic type should hold, and we expect that if $\underline{u}_1, \ldots \underline{u}_m$ are couples of algebraic numbers in $\mathbb{T}$ close enough to $(0,0) \in \mathbb{C}^2$, such that for two distinct irrationals $w_1, w_2 > 0$ two relations:

$$\sum_{i=1}^{m} c_i f_{w_j}(\underline{u}_i) = l_j$$

hold for rational numbers $c_1, \ldots, c_m$ not all zero, $l_j \in \overline{\mathbb{Q}}$ and $j = 1, 2$, then $l_1 = l_2 = 0$ and the two relations both arise from a single relation of generic type. However, we do not know how to prove this even when the second coordinate of the points involved is equal to 1.

# References


[He] E. Hecke. *Analytische Funktionen und algebraische Zahlen. Erster Teil,* Abh. Math. Sem. Hamburg Univ. 1 (1921), 102-126.

[Ku] K. K. Kubota. *On the Algebraic Independence of Holomorphic Solutions of Certain Functional Equations and their Values,* Math. Ann. 227, pp. 9-50 (1977).

[Lo-Po1] J. H. Loxton & A. J. van der Poorten. *Arithmetic properties of certain functions in several variables,* J. Number Theory 9 pp. 87-106 (1977).

[Lo-Po2] J. H. Loxton & A. J. van der Poorten. *Arithmetic properties of certain functions in several variables II,* J. Australian Math. Soc. A24 pp. 393-408 (1977).

[Lo-Po3] J. H. Loxton & A. J. van der Poorten. *Algebraic independence properties of the Fredholm series,* J. Australian Math. Soc. A29 pp. 31-45 (1978).

[Mah] K. Mahler. *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen,* Math. Annalen 101 (1929), 342-366.

[Mas1] D. W. Masser. *A Vanishing Theorem for Power Series,* Invent. Math. 67, (1982), pp. 275-296.

[Mas2] D. W. Masser. *Algebraic independence properties of the Hecke-Mahler series,* Quart. J. Math. Oxford (2), 50 (1999), 207-230.

[Ni] K. Nishioka. *Mahler functions and transcendence.* Springer Lecture Notes in Mathematics. 1631, (1996).

[Pel1] F. Pellarin. *On the arithmetic properties of complex values of Hecke-Mahler series I,* to appear in Ann. Scuola Norm. Sup. Pisa, Cl. di Sci. (5) Vol. V pp. 1-46 (2006). http://www.math.unicaen.fr/~pellarin/HeMa1.pdf.

[Pel2] F. Pellarin. *Propriétés d'indépendance algébrique de valeurs de séries de Hecke–Mahler,* C. R. Acad. Sci. Paris, Ser. I. 340, pp. 861-866 (2005).

[Per] O. Perron. *Die Lehre von den Kettenbrüchen,* Chelsea Publishing Company, New York (1957).

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