

Universal Gauss-Thakur sums and L -series ^{*} [†]

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Abstract. We prove that the maximal abelian extension tamely ramified at infinity of the rational function field over \mathbb{F}_q is generated by the values at the points in the algebraic closure of \mathbb{F}_q of the higher derivatives of the so-called Anderson and Thakur function ω . We deduce a similar property for the special values of the higher derivatives of a new kind of L -series introduced by the second author.

1 Results

In virtue of the Kronecker-Weber Theorem, the maximal abelian extension of \mathbb{Q} in \mathbb{C} can be generated by the values of the exponential function at the rational multiples of $\pi\sqrt{-1}$. In [9], Hayes proves a similar property for the Carlitz exponential \exp_C .

Let \mathbb{F}_q be the finite field with q elements and let K be the rational function field over \mathbb{F}_q . Let us choose a generator θ of K and let us denote by \mathbb{C}_∞ the completion of an algebraic closure of the local field $K_\infty = \mathbb{F}_q((1/\theta))$.

The *Carlitz exponential* \exp_C (Goss, [8, §3.2]) is a surjective, entire, \mathbb{F}_q -linear map $\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ of kernel $\tilde{\pi}A$, where $A = \mathbb{F}_q[\theta]$ and where the period

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty \quad (1)$$

is defined up to the multiplication by an element of $\mathbb{F}_q^\times = \mathbb{F}_q \setminus \{0\}$.

Let \mathbb{F}_q^{ac} be the algebraic closure of \mathbb{F}_q in \mathbb{C}_∞ . Hayes result [9, Theorem 7.1] yields that the maximal abelian extension $E \subset \mathbb{C}_\infty$ of K tamely ramified at the infinity place is the compositum in \mathbb{C}_∞ of \mathbb{F}_q^{ac} , and the subfield of \mathbb{C}_∞ generated over K by the images of \exp_C at the elements of $\tilde{\pi}K$.

The aim of this paper is to exhibit new ways to generate the extension E . Let us denote by D the disk $\{z \in \mathbb{C}_\infty; |z| \leq 1\}$. The *special function of Anderson and Thakur* ω , introduced in [4, Proof of Lemma 2.5.4], can be defined for all $x \in D$ by the convergent infinite product

$$\omega(x) = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{x}{\theta^{q^i}}\right)^{-1}, \quad (2)$$

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where we choose the same branch of the $(q-1)$ -th root of $-\theta$ as in the product (1).

This function already played a singular arithmetic role in previous literature. For instance, the function

$$\Omega = \frac{1}{(t-\theta)\omega},$$

entire, determines a *rigid analytic trivialization* of the Carlitz (dual) t -motive (following Anderson, Brownawell and Papanikolas in [3]).

The *higher derivatives* $D_n(\omega)$ of the function ω (see §2) all determine rigid analytic functions $D \rightarrow \mathbb{C}_\infty$ and we can consider the values $D_n(\omega)(\zeta) \in \mathbb{C}_\infty$ for all $n \geq 0$ and $\zeta \in \mathbb{F}_q^{ac}$. It turns out that these are algebraic elements over K (see Proposition 2.1).

For each monic irreducible polynomial \mathfrak{p} of A (abridged to *prime* of A in all the following), we choose a root $\zeta_{\mathfrak{p}} \in \mathbb{F}_q^{ac}$. We shall prove:

Theorem 1.1 *The maximal abelian extension E of K tamely ramified at the infinity place is generated over \mathbb{F}_q by the elements $D_n(\omega)(\zeta_{\mathfrak{p}})$ for all $n \geq 0$ and for every prime \mathfrak{p} of A .*

In [12], the second author introduced a class of L -series which analytically interpolate the Carlitz zeta values and the special values of Dirichlet-Goss L -series. The simplest of these functions is defined by the eulerian product

$$\mathfrak{L}(x) = \prod_{\mathfrak{p}} \left(1 - \frac{\mathfrak{p}(x)}{\mathfrak{p}}\right)^{-1}.$$

The above product runs over the primes of A , and for a polynomial $a = \sum_{i=0}^d a_i \theta^i \in A = \mathbb{F}_q[\theta]$ and $x \in \mathbb{C}_\infty$, we have denoted by $a(x)$ the element $\sum_{i=0}^d a_i x^i \in \mathbb{F}_q[x]$. We have convergence for all $x \in D$. In [12, Theorem 1], the functional identity

$$\mathfrak{L} = -\tilde{\pi}\Omega \tag{3}$$

is proved, hence featuring ω as a “gamma factor”. This allows for the analytic study of trivial zeroes and special values of \mathfrak{L} . By (3) and Theorem 1.1 we obtain the following result which appears to be in the spirit of Stark conjectures (see Tate’s book [14]):

Corollary 1.2 *The field E is generated over \mathbb{F}_q by the elements $\tilde{\pi}^{-1}D_n(\mathfrak{L})(\zeta_{\mathfrak{p}})$ for all $n \geq 0$ and for all prime \mathfrak{p} .*

The paper is organized as follows. In §2 we present the properties of ω that we develop to prove our results. In particular, we show an *analytic formula* (Proposition 2.6) and, in Theorem 2.9, we give an interpretation of ω as a “universal Gauss-Thakur sum”; this is the result which originally motivated our investigation. Formal analogies between the classical Gauss sums and Euler’s gamma function also suggested this approach.

In §3, the proofs of Theorems 1.1 and Corollary 1.2 are deduced from an identity of fields at the level of finite extensions of K (Theorem 3.3). In §3.1 we examine the compatibility of our constructions with class field theory. We end with a few remarks about our methods. We point out that, for the time being, our constructions do not seem to have appropriate analogues in the class field theory for the rational field \mathbb{Q} .

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2 The function of Anderson and Thakur

Let $|\cdot|$ be the absolute value of \mathbb{C}_∞ normalized by $|\theta| = q$. Let $\|\cdot\|$ be the Gauss absolute value over $\mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$ determined by $\|c \otimes 1\| = |c|$ for all $c \in \mathbb{C}_\infty$. We denote by \mathbb{T} the completion of $\mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$ for this absolute value (we identify \mathbb{C}_∞ with its image by the \mathbb{C}_∞ -algebra embedding $\mathbb{C}_\infty \rightarrow \mathbb{T}$ defined by $c \mapsto c \otimes 1$). This is the standard one-dimensional Tate algebra over \mathbb{C}_∞ (see [7] for an account of the theory of these algebras). If we denote by t the element $1 \otimes \theta \in \mathbb{T}$, then every element f of \mathbb{T} can be represented by a series

$$f = \sum_{k \geq 0} f_k t^k,$$

with $f_k \in \mathbb{C}_\infty$ for all $k \geq 0$ and with $\lim_{k \rightarrow \infty} f_k = 0$. In particular, for all $\zeta \in \mathbb{C}_\infty$ such that $|\zeta| \leq 1$, the series

$$f(\zeta) := \sum_{k \geq 0} f_k \zeta^k$$

converges in \mathbb{C}_∞ .

Let $\tau : \mathbb{T} \rightarrow \mathbb{T}$ be the unique $\mathbb{F}_q[t]$ -linear automorphism extending the \mathbb{F}_q -automorphism $c \mapsto c^q$ of \mathbb{C}_∞ . For all $f \in \mathbb{T}$, we have

$$\|\tau(f)\| = \|f\|^q.$$

We denote by $\mathbb{T}[\tau]$ the skew polynomial ring whose elements are the finite sums $a_0 + a_1 \tau + \cdots + a_n \tau^n$ with $a_0, \dots, a_n \in \mathbb{T}$, where the sum is the usual one, and where the product is uniquely determined by the rule

$$\tau f = \tau(f) \tau$$

for $f \in \mathbb{T}$. If $\alpha = a_0 + \cdots + a_n \tau^n \in \mathbb{T}[\tau]$ and $f \in \mathbb{T}$, the *evaluation* $\alpha(f)$ of α at f is the element $a_0 f + \cdots + a_n \tau^n(f) \in \mathbb{T}$.

For all $n \geq 0$, we denote by D_n the continuous \mathbb{C}_∞ -linear endomorphism of \mathbb{T} determined by the relations

$$D_m(t^n) = \binom{n}{m} t^{n-m}$$

for $m \geq 0$, where $\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!} \in \mathbb{F}_q$. Then, $(D_n)_{n \geq 0}$ is an *iterative higher derivative* in the sense of Matsumura, [10, §27]. For all $n \geq 0$ and for all $f \in \mathbb{T}$, $D_n(f) \in \mathbb{T}$. We note that for all $n \geq 0$, the operators D_n and τ commute.

2.1 Values of ω at roots of unity

In this subsection we briefly discuss the algebraicity of ω and its higher derivatives at roots of unity.

Proposition 2.1 *For all $n \geq 0$ and $\zeta \in \mathbb{F}_q^{ac}$, $D_n(\omega)(\zeta)$ is algebraic over K .*

Proof. From the definition (2) of ω , we see that

$$\tau(\omega) = (t - \theta)\omega. \tag{4}$$

Hence, we have that

$$\tau^d(\omega) = b_d \omega \tag{5}$$

where $b_d = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}})$.

Let $\zeta \in \mathbb{F}_q^{ac}$ be of degree d over \mathbb{F}_q . Since for all $f \in \mathbb{T}$ we have $f(\zeta)^{q^d} = \tau^d(f)(\zeta)$, we have that $\omega(\zeta)^{q^d} = (\tau^d(\omega))(\zeta)$ and $X = \omega(\zeta)$ is a root of the polynomial equation

$$X^{q^d-1} = b_d(\zeta). \quad (6)$$

In particular, $\omega(\zeta)$ is algebraic over K (cf. Theorem 2.9). Now, let $n > 0$ be an integer. Since the operators D_n and τ commute, we infer from (5) that

$$\tau^d(D_n(\omega)) = b_d D_n(\omega) + \sum_{i=1}^n D_i(b_d) D_{n-i}(\omega)$$

(since $(D_n)_{n \geq 0}$ is a higher derivative, it satisfies Leibniz rule). Specializing at $t = \zeta$ we obtain that $X = D_n(\omega)(\zeta)$ is a root of a polynomial equation:

$$X^{q^d} = b_d(\zeta)X + \xi_n, \quad (7)$$

where

$$\xi_n = \sum_{i=1}^n D_i(g_d)(\zeta) D_{n-i}(\omega)(\zeta), \quad (8)$$

that is, a linear combination of $\omega(\zeta), D_1(\omega)(\zeta), \dots, D_{n-1}(\omega)(\zeta)$ with coefficients in $A[\zeta]$. By induction on n , we conclude the proof. \square

Remark 2.2 From [12, Corollaries 5, 10] one can prove that $\zeta \in \mathbb{C}_\infty \setminus \{\theta^{q^k}; k \geq 0\}$ and $\omega(\zeta)$ are simultaneously algebraic if and only if $\zeta \in \mathbb{F}_q^{ac}$.

2.2 The Carlitz module over the Tate algebra

The next definition is borrowed from [6]; it is a particular case of *Drinfeld module over Tate algebras* studied there.

Definition 2.3 The *Carlitz module over \mathbb{T}* , denoted by $C(\mathbb{T})$, is the $\mathbb{F}_q[t]$ -module \mathbb{T} together with the unique $A[t]$ -module structure for which the action of θ is given by the evaluation of the skew polynomial $\theta + \tau$.

For $a \in A[t]$, $f \in \mathbb{T}$, we denote by $C_a(f)$ the multiplication of f by a for this module structure. Here is an example which illustrates this action: if $f \in \mathbb{T}$, we have $C_\theta(f) = \theta f + \tau(f)$.

Definition 2.4 The *Carlitz exponential function* over \mathbb{T} is the continuous, open, $\mathbb{F}_q[t]$ -linear endomorphism \exp_C of \mathbb{T} such that, for $f \in \mathbb{T}$,

$$\exp_C(f) = \sum_{i \geq 0} \frac{\tau^i(f)}{d_i},$$

where $d_0 = 1$ and $d_i = (\theta^{q^i} - \theta)d_{i-1}^q$.

Since the restriction of \exp_C to $\mathbb{C}_\infty \subset \mathbb{T}$ gives the usual Carlitz exponential function from \mathbb{C}_∞ to \mathbb{C}_∞ , which is entire and surjective, the function $\exp_C : \mathbb{T} \rightarrow \mathbb{T}$ is itself surjective. Moreover, for all $a \in A[t]$ and $f \in \mathbb{T}$, we have the identity

$$C_a(\exp_C(f)) = \exp_C(af).$$

Now, \exp_C induces an isometric $\mathbb{F}_q[t]$ -linear automorphism of the $\mathbb{F}_q[t]$ -module $\{f \in \mathbb{T}; \|f\| < q^{\frac{q}{q-1}}\}$. Hence, if an element f belongs to the kernel of \exp_C , it must be a polynomial in t . The knowledge of the kernel of $\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ finally yields (cf. [6, §3.2.1]):

Lemma 2.5 *The Carlitz exponential function gives rise to the exact sequence of $A[t]$ -modules:*

$$0 \rightarrow \tilde{\pi}A[t] \rightarrow \mathbb{T} \rightarrow C(\mathbb{T}) \rightarrow 0.$$

It is easy to show that the kernel of the $\mathbb{F}_q[t]$ -linear endomorphism $C_{t-\theta} = t - C_\theta$ of \mathbb{T} is the free $\mathbb{F}_q[t]$ -module of rank one generated by $\mu = \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right)$. In particular, this element of \mathbb{T} is solution of the τ -difference equation (4). Since also ω is a solution of that equation and both ω and μ are units in \mathbb{T} , this implies that $\omega \in \mathbb{F}_q^\times \mu$. Comparing the values of both functions at $t = 0$, we obtain that $\omega = \mu$, that is (cf. [12, §4]),

$$\omega = \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right). \quad (9)$$

But for all $n \geq 0$ the operators D_n and τ commute, and we also have

$$D_n(\omega) = \exp_C\left(\frac{\tilde{\pi}}{(\theta-t)^{n+1}}\right). \quad (10)$$

2.3 An analytic identity

In order to prove Theorem 1.1, we need to slightly generalize the identity (10).

For $\zeta \in \mathbb{F}_{q^d}$ of degree d , we introduce the series:

$$\omega_\zeta = \omega(t + \zeta) = \sum_{j \geq 0} c_j^\zeta t^j \in \mathbb{T}$$

where

$$c_j^\zeta = D_j(\omega)(\zeta) = \sum_{i \geq j} c_i^0 \binom{i}{j} \zeta^{i-j}$$

with $c_i^0 = \exp_C(\tilde{\pi}/\mathfrak{p}^{n+1}) \in K_\infty(\tilde{\pi})$.

Proposition 2.6 (Analytic identity) *Let \mathfrak{p} be a prime of A of degree d , let $\zeta_{\mathfrak{p}}$ be one of its roots in \mathbb{F}_{q^d} . Let $P(t) \in A[t]$ be defined by $P(t) = \mathfrak{p}(\theta - t)$. Then, for all $n \geq 0$, there exists a polynomial $B_n \in A[t] \setminus PA[t]$ such that the following identity in \mathbb{T} holds:*

$$D_n(\omega_{\zeta_{\mathfrak{p}}}) = - \sum_{a \in A(d)} \frac{C_a(\exp_C(\frac{\tilde{\pi}B_n}{P^{n+1}}))}{a(t + \zeta_{\mathfrak{p}})} - \sum_{j=1}^{d-1} M_j D_n(\omega_{\zeta_{\mathfrak{p}}^{q^j}}) - \sum_{k=0}^{n-1} \sum_{j=0}^{d-1} N_{n-k,j} D_k\left(\omega_{\zeta_{\mathfrak{p}}^{q^j}}\right), \quad (11)$$

where $A(d)$ denotes the set of non-zero polynomials of A of degree $\leq d-1$, and where

$$M_j = -\prod_{i=1}^{d-1} \frac{t - t^{q^i} + \zeta_{\mathfrak{p}}^{q^j} - \zeta_{\mathfrak{p}}^{q^i}}{t - t^{q^i} + \zeta_{\mathfrak{p}} - \zeta_{\mathfrak{p}}^{q^i}}, \quad N_{i,j} = \sum_{a \in A(d)} \frac{D_i(a(t + \zeta_{\mathfrak{p}}^{q^j}))}{a(t + \zeta_{\mathfrak{p}})}.$$

Proof. Let ζ_1, \dots, ζ_d be the zeros of \mathfrak{p} , and let us set $\zeta_{\mathfrak{p}} = \zeta_1$. It is easy to show that there exists a polynomial $B_n \in A[t] \setminus PA[t]$ such that

$$\frac{B_n}{P^{n+1}} = \sum_{i=1}^d \frac{1}{(\theta - t - \zeta_i)^{n+1}}.$$

We now apply the operator $\exp_C(\tilde{\pi} \cdot)$ to both sides (we note that all the terms involved are elements of \mathbb{T}). We get the following identity in \mathbb{T} :

$$\exp_C \left(\frac{\tilde{\pi} B_n}{P^{n+1}} \right) = \exp_C \left(\sum_{i=1}^d \frac{\tilde{\pi}}{(\theta - t - \zeta_i)^{n+1}} \right).$$

We claim that if f is an element of \mathbb{T} , then

$$\sum_{i=1}^d \exp_C(f)(t + \zeta_i) = \exp_C \left(\sum_{i=1}^d f(t + \zeta_i) \right).$$

This comes from the fact that for all $k \geq 0$,

$$\sum_{j=1}^d \tau^k(f)(t + \zeta_j) = \tau^k \left(\sum_{j=1}^d f(t + \zeta_j) \right).$$

Applying the claim, we get the identity

$$\sum_{i=1}^d D_n(\omega_{\zeta_i}) = \exp_C \left(\frac{\tilde{\pi} B_n}{P^{n+1}} \right). \quad (12)$$

Let $\rho \in \text{Gal}(\mathbb{F}_{q^d} K_{\infty}(\tilde{\pi}) / K_{\infty}(\tilde{\pi}))$ be the unique element of order d such that $\rho(\zeta) = \zeta^q$ for all $\zeta \in \mathbb{F}_{q^d}$.

We consider the \mathbb{F}_{q^d} -linear endomorphism

$$\tilde{\tau} = \rho^{-1} \circ \tau : \mathbb{F}_{q^d} K_{\infty}(\tilde{\pi}) \rightarrow \mathbb{F}_{q^d} K_{\infty}(\tilde{\pi}), \quad (13)$$

which satisfies $|\tilde{\tau}(x)| = |x|^q$ for all $x \in \mathbb{F}_{q^d} K_{\infty}(\tilde{\pi})$. Let us consider the $\mathbb{F}_{q^d} K_{\infty}(\tilde{\pi})$ -algebra $\mathbb{T}_{d,\infty}$ whose elements are the series $\sum_i c_i t^i$ of \mathbb{T} such that the coefficients c_i belong to $\mathbb{F}_{q^d} K_{\infty}(\tilde{\pi})$.

We extend $\tilde{\tau}$ to an $\mathbb{F}_{q^d}[t]$ -linear endomorphism of $\mathbb{T}_{d,\infty}$. We endow $\mathbb{T}_{d,\infty}$ with the unique structure of $\mathbb{F}_{q^d}[\theta]$ -module $\tilde{C}(\mathbb{T}_{d,\infty})$ determined by $\tilde{C}_{\theta}(x) = \theta x + \tilde{\tau}(x)$ for $x \in \mathbb{T}_{d,\infty}$ (here, $\tilde{C}_a(x)$ denotes the (left) multiplication of x by $a \in \mathbb{F}_{q^d}[\theta]$ for this module structure). If $a \in A$ and $C_a = \sum_{i \geq 0} (a)_i \tau^i$ with $(a)_i \in A$, then \tilde{C}_a can be viewed as the skew polynomial $\sum_{i \geq 0} (a)_i \tilde{\tau}^i$, in powers of $\tilde{\tau}$, which can be evaluated at elements of $\mathbb{T}_{d,\infty}$; if $f \in \mathbb{T}_{d,\infty}$, $\tilde{C}_a(f) = \sum_{i \geq 0} (a)_i \tilde{\tau}^i(f)$.

It is clear that $\omega_\zeta \in \mathbb{T}_{d,\infty}$ (here, $\zeta = \zeta_1 = \zeta_{\mathfrak{p}}$). We observe that

$$\tilde{\tau}(\omega_\zeta) = (t + \zeta - \theta)\omega_\zeta. \quad (14)$$

This implies that, for all $a \in \mathbb{F}_{q^d}[\theta]$,

$$\tilde{C}_a(\omega_\zeta) = a(t + \zeta)\omega_\zeta. \quad (15)$$

Since the operators D_n ($n \geq 0$) and $\tilde{\tau}$ commute over $\mathbb{T}_{d,\infty}$, we have that $\tilde{C}_a \circ D_n = D_n \circ \tilde{C}_a$ and we infer that

$$\tilde{C}_a(D_n(\omega_\zeta)) = \sum_{i=0}^n D_i(\omega_\zeta) D_{n-i}(a(t + \zeta)). \quad (16)$$

Moreover, if Tr denotes the trace map of $\mathbb{F}_{q^d}K_\infty(\tilde{\pi})$ over $K_\infty(\tilde{\pi})$, one sees that, over $\mathbb{F}_{q^d}K_\infty(\tilde{\pi})$:

$$\text{Tr} \circ \tilde{C}_a = C_a \circ \text{Tr}, \quad a \in A.$$

Applying C_a to both left- and right-hand sides of (12), we get the identity

$$\sum_{i=1}^d a(t + \zeta_i) D_n(\omega_{\zeta_i}) + \sum_{k=0}^{n-1} \sum_{i=1}^d D_{n-k}(a(t + \zeta_i)) D_k(\omega_{\zeta_i}) = C_a \left(\exp_C \left(\frac{\tilde{\pi} B_n}{P^{n+1}} \right) \right). \quad (17)$$

In the above identity, we divide by $a(t + \zeta_{\mathfrak{p}})$ and we sum over $A(d)$. The coefficient of $D_n(\omega_{\zeta_{\mathfrak{p}}})$ that we obtain is -1 . The coefficient of $D_n(\omega_{\zeta_i})$ with $i = 2, \dots, d$ is equal to

$$\sum_{a \in A(d)} \frac{a(t + \zeta_i)}{a(t + \zeta_{\mathfrak{p}})}.$$

To factorize it, we use the following formula in $\mathbb{F}_q(X, Y)$ whose proof is easy and left to the reader (X, Y are two indeterminates):

$$\sum_{a \in A(d)} \frac{a(X)}{a(Y)} = - \prod_{i=1}^{d-1} \frac{X - Y^{q^i}}{Y - Y^{q^i}}.$$

Replacing X with $t + \zeta_{\mathfrak{p}}^{q^j}$ and Y with $t + \zeta_{\mathfrak{p}}$ we obtain M_j . The proposition follows at once. \square

Remark 2.7 It may be interesting to note that, for $\zeta \in \mathbb{F}_{q^d}$, $\omega_\zeta = \exp_{\tilde{C}} \left(\frac{\tilde{\pi}}{\theta - t - \zeta} \right)$, where

$$\exp_{\tilde{C}}(f) = \sum_{i \geq 0} D_i^{-1} \tilde{\tau}^i(f) \in \mathbb{T}_{d,\infty}$$

for $f \in \mathbb{T}_{d,\infty}$, but we will not make use of this property so we omit the proof.

2.4 Gauss-Thakur sums

We recall that Thakur established several analogues of classical results about Gauss sums such as Stickelberger factorization theorem and Gross-Koblitz formulas and others (see for example [15, 17]).

Let \mathfrak{p} be a prime of $A = \mathbb{F}_q[\theta]$ of degree d . We write

$$\lambda_{\mathfrak{p}} = \exp_C \left(\frac{\tilde{\pi}}{\mathfrak{p}} \right).$$

We denote by $K_{\mathfrak{p}}$ the \mathfrak{p} -th cyclotomic function field extension $K(\lambda_{\mathfrak{p}})$ of K in \mathbb{C}_{∞} . We refer the reader to [13, Chapter 12] for the basic properties of cyclotomic function fields. We recall here that the integral closure $\mathcal{O}_{K_{\mathfrak{p}}}$ of A in $K_{\mathfrak{p}}$ equals the ring $A[\lambda_{\mathfrak{p}}]$.

The extension $K_{\mathfrak{p}}/K$ is cyclic of degree $q^d - 1$, ramified in \mathfrak{p} and θ^{-1} . It is in fact totally ramified in \mathfrak{p} and the decomposition group at θ^{-1} is isomorphic to the inertia group, therefore isomorphic to \mathbb{F}_q^{\times} . We denote by $\Delta_{\mathfrak{p}}$ the Galois group $\text{Gal}(K_{\mathfrak{p}}/K)$. Since the constant subfield of $K_{\mathfrak{p}}$ is \mathbb{F}_q , $\Delta_{\mathfrak{p}}$ is canonically isomorphic to the Galois group of the extension $\mathbb{F}_{q^d}K_{\mathfrak{p}}/\mathbb{F}_{q^d}K$ (if F, G are subfields of \mathbb{C}_{∞} , we always denote by FG their compositum).

There is a unique isomorphism (Artin symbol, [8, Proposition 7.5.4])

$$\sigma : (A/\mathfrak{p}A)^{\times} \rightarrow \Delta_{\mathfrak{p}}$$

such that

$$\sigma_a(\lambda_{\mathfrak{p}}) = C_a(\lambda_{\mathfrak{p}}).$$

Once a choice of a root $\zeta_{\mathfrak{p}}$ of \mathfrak{p} is made, the *Teichmüller character* (see [8, Section 8.11]) induces a unique group isomorphism

$$\vartheta_{\mathfrak{p}} : \Delta_{\mathfrak{p}} \rightarrow \mathbb{F}_{q^d}^{\times},$$

defined in the following way. If $\delta = \sigma_a \in \Delta_{\mathfrak{p}}$ for some $a \in A$, then

$$\vartheta_{\mathfrak{p}}(\delta) = a(\zeta_{\mathfrak{p}}). \quad (18)$$

For any finite abelian group G , we shall write \hat{G} for the group $\text{Hom}(G, (\mathbb{F}_q^{\text{ac}})^{\times})$. In particular, $\vartheta_{\mathfrak{p}} \in \hat{\Delta}_{\mathfrak{p}}$.

Definition 2.8 For $j = 0, \dots, d-1$, the *basic Gauss-Thakur sum* $g(\vartheta_{\mathfrak{p}}^{q^j})$ is the element of \mathbb{C}_{∞} defined by:

$$g(\vartheta_{\mathfrak{p}}^{q^j}) = - \sum_{\delta \in \Delta_{\mathfrak{p}}} \vartheta_{\mathfrak{p}}(\delta^{-1})^{q^j} \delta(\lambda_{\mathfrak{p}}) \in \mathbb{F}_{q^d}[\theta][\lambda_{\mathfrak{p}}].$$

The same sum is denoted by g_j in [8, 15]. For our purposes, we will only need to work with $g(\vartheta_{\mathfrak{p}})$.

By [15, Theorem I], $g(\vartheta_{\mathfrak{p}})$ is non-zero and for all $\delta \in \Delta_{\mathfrak{p}}$, we have:

$$\delta(g(\vartheta_{\mathfrak{p}})) = \vartheta_{\mathfrak{p}}(\delta)g(\vartheta_{\mathfrak{p}}). \quad (19)$$

In the next theorem, \mathfrak{p}' denotes the derivative of \mathfrak{p} with respect to θ .

Theorem 2.9 *We have the following identity:*

$$g(\vartheta_{\mathfrak{p}}) = \mathfrak{p}'(\zeta_{\mathfrak{p}})^{-1} \omega(\zeta_{\mathfrak{p}}).$$

Proof. We use some ideas of the proof of Thakur's result [15, Theorem II]. Let

$$\text{sgn} : \mathbb{F}_{q^d}((1/\theta))^\times \rightarrow \mathbb{F}_{q^d}^\times$$

be the unique group homomorphism (sign function) such that $\text{sgn}(\theta) = 1$ and inducing the identity map over $\mathbb{F}_{q^d}^\times$. Then by [17, Theorem 2.3], we have:

$$\text{sgn}(g(\vartheta_{\mathfrak{p}})\tilde{\pi}^{-1}) = \vartheta_{\mathfrak{p}}(\sigma_{\mathfrak{p}'})^{-1} = \mathfrak{p}'(\zeta_{\mathfrak{p}})^{-1}.$$

In virtue of (1), $\text{sgn}(\lambda_\theta \tilde{\pi}^{-1}) = 1$. Therefore: $\text{sgn}(g(\vartheta_{\mathfrak{p}})(-\theta)^{-1/(q-1)}) = \mathfrak{p}'(\zeta_{\mathfrak{p}})^{-1}$. Now, by [15, Theorem IV], we have $|g(\vartheta_{\mathfrak{p}})(-\theta)^{-\frac{1}{q-1}}| = 1$, so that

$$\left| \frac{g(\vartheta_{\mathfrak{p}})}{(-\theta)^{\frac{1}{q-1}}} - \mathfrak{p}'(\zeta)^{-1} \right| < 1. \quad (20)$$

By the proof of [15, Theorem II] (see Equation (3) there) or by the fact that $\tau(x) = C_\theta(x) - \theta x$ for all $x \in \mathbb{T}$, we have that $g(\vartheta_{\mathfrak{p}}^{q^{j-1}})^q = (\zeta_{\mathfrak{p}} - \theta)g(\vartheta_{\mathfrak{p}}^{q^j})$. Moreover, $\rho(g(\vartheta_{\mathfrak{p}}^{q^{j-1}})) = g(\vartheta_{\mathfrak{p}}^{q^j})$ where ρ is defined just before (13). Therefore, we get, with $\tilde{\tau}$ defined as in (13):

$$\tilde{\tau}(g(\vartheta_{\mathfrak{p}})) = (\zeta_{\mathfrak{p}} - \theta)g(\vartheta_{\mathfrak{p}}).$$

Iterating, this implies that for all $n \geq 1$,

$$\tilde{\tau}^n \left(\frac{g(\vartheta_{\mathfrak{p}})}{(-\theta)^{\frac{1}{q-1}}} \right) = (-\theta)^{-\frac{1}{q-1}} \prod_{i=0}^{n-1} \left(1 - \frac{\zeta_{\mathfrak{p}}}{\theta^{q^i}} \right) g(\vartheta_{\mathfrak{p}}).$$

But:

$$\lim_{n \rightarrow \infty} (-\theta)^{-\frac{1}{q-1}} \prod_{i=0}^{n-1} \left(1 - \frac{\zeta_{\mathfrak{p}}}{\theta^{q^i}} \right) = \omega(\zeta_{\mathfrak{p}})^{-1},$$

and, thanks to (20),

$$\lim_{n \rightarrow \infty} \tilde{\tau}^n \left(\frac{g(\vartheta_{\mathfrak{p}})}{(-\theta)^{\frac{1}{q-1}}} \right) = \mathfrak{p}'(\zeta_{\mathfrak{p}})^{-1}.$$

The Theorem follows. □

Remark 2.10 It can be proved that Theorem 2.9 actually implies the functional identity (3). The proof runs along essentially the same lines of the proof of [5, Theorem 1] and we omit the details.

3 Proofs of the main results

Let \mathfrak{p} be a prime of A of degree d . We set

$$\lambda_{\mathfrak{p}^{n+1}} = \exp_C \left(\frac{\tilde{\pi}}{\mathfrak{p}^{n+1}} \right), \quad n \geq 0$$

and we denote by $K_{\mathfrak{p}^{n+1}}$ the field $K(\lambda_{\mathfrak{p}^{n+1}})$. We recall [13, Proposition 12.7] that the extension $K_{\mathfrak{p}^{n+1}}/K$ is a Galois extension of degree $e_n = q^{dn}(q^d - 1)$, unramified at each prime distinct from

\mathfrak{p} and totally ramified at \mathfrak{p} . Moreover, $\text{Gal}(K_{\mathfrak{p}^{n+1}}/K)$ is isomorphic to $(A/\mathfrak{p}^{n+1}A)^\times$. The prime ideal above \mathfrak{p} is $\lambda_{\mathfrak{p}^{n+1}}A[\lambda_{\mathfrak{p}^{n+1}}]$, ($A[\lambda_{\mathfrak{p}^{n+1}}]$ is the ring of integers of $K_{\mathfrak{p}^{n+1}}$). Let $\zeta_{\mathfrak{p}} \in \mathbb{F}_{q^d}$ be any root of \mathfrak{p} . For all $n \geq 0$ and $j = 0, \dots, d-1$, let us denote by $Q_{j,n+1}$ the prime ideal of $\mathbb{F}_{q^d}[\theta][\lambda_{\mathfrak{p}^{n+1}}]$ which lies above $\theta - \zeta_{\mathfrak{p}}^{q^j}$. Let $v_{j,n} : \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}} \rightarrow \mathbb{Z}$ be the valuation associated to the ideal $Q_{j,n+1}$, normalized by $v_{j,n}(\zeta_{\mathfrak{p}}^{q^j} - \theta) = e_n$, so that we have $v_{j,n} = q^d v_{j,n-1}$ over $\mathbb{F}_{q^d}K_{\mathfrak{p}^n}$ for all $n > 0$.

Lemma 3.1 *Let \mathfrak{p} be a monic irreducible element of A of degree d and let n be an integer ≥ 0 . Let $\mu \in \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$ be such that $v_{0,n}(\mu) = 1$ and $v_{j,n}(\mu) > 1$ for $j = 1, \dots, d-1$. Then $K(\mu) = \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$.*

Proof. Since $\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K$ is an abelian extension, $K(\mu)/K$ is also an abelian extension. Let $j \in \{1, \dots, d-1\}$. Then, there exists $\sigma \in \text{Gal}(\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K)$ such that $v_{j,n}(\sigma(\mu)) = 1$ and $v_{k,n}(\sigma(\mu)) > 1$ for $k \in \{0, \dots, d-1\}$, such that $k \neq j$. Thus $\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K(\mu)$ is unramified at the primes above \mathfrak{p} and for distinct $k, j \in \{0, \dots, d-1\}$, $v_{j,n}|_{K(\mu)} \neq v_{k,n}|_{K(\mu)}$. This implies $K(\mu) = \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$. \square

By the fact that $\zeta_{\mathfrak{p}} - \theta^{q^j} = (\zeta_{\mathfrak{p}}^{q^{d-j}} - \theta)^{q^j}$, we notice that:

$$v_{0,n}(b_d(\zeta)) = e_n, \quad v_{d-k,n}(b_d(\zeta)) = q^k e_n, \quad k = 1, \dots, d-1. \quad (21)$$

Corollary 3.2 *We have that*

$$K(g(\vartheta_{\mathfrak{p}})) = K(\omega(\zeta_{\mathfrak{p}})) = \mathbb{F}_{q^d}K_{\mathfrak{p}}.$$

Proof. This follows from Theorem 2.9, [15, Theorem IV], and Lemma 3.1. \square

Theorem 1.1 will be deduced from the next generalization of Corollary 3.2.

Theorem 3.3 *For all $n \geq 0$, we have*

$$K(D_n(\omega)(\zeta_{\mathfrak{p}})) = \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}.$$

Proof. The case $n = 0$ is dealt in Corollary 3.2. Let us now consider the case $n \geq 0$. We first replace $t = 0$ in the identity (11) of Proposition 2.6. Since the fractions M_j vanish at $t = 0$ we get, for some $b_n \in A \setminus \mathfrak{p}A$ and with our choice of root $\zeta_{\mathfrak{p}}$ of \mathfrak{p} :

$$D_n(\omega)(\zeta_{\mathfrak{p}}) = - \sum_{a \in A(d)} a(\zeta_{\mathfrak{p}})^{-1} C_{ab_n}(\lambda_{\mathfrak{p}^{n+1}}) - \sum_{k=0}^{n-1} \sum_{j=0}^{d-1} N_{n-k,j}(0) D_k(\omega)(\zeta_{\mathfrak{p}}^{q^j}). \quad (22)$$

We show by induction over $n > 0$ that $D_n(\omega)(\zeta_{\mathfrak{p}}) \in \mathbb{F}_{q^d}[\theta][\lambda_{\mathfrak{p}^{n+1}}]$ and that

$$v_{0,n}(D_n(\omega)(\zeta_{\mathfrak{p}})) = 1, \text{ and } v_{d-k,n}(D_n(\omega)(\zeta_{\mathfrak{p}})) = q^k, \quad k = 1, \dots, d-1. \quad (23)$$

Let us assume that for an integer $n > 0$, we have proved this property for $i = 0, \dots, n-1$. By (7) we obtain that $D_n(\omega)(\zeta_{\mathfrak{p}})$ is an algebraic integer. The identity (22), the fact that the extension $\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K$ is abelian, and the induction hypothesis, tell us that $K(D_n(\omega)(\zeta_{\mathfrak{p}})) \subset \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$. We observe on the way, by (7) and induction, that

$$\sigma(K(D_n(\omega)(\zeta_{\mathfrak{p}}))) = K(D_n(\omega)(\sigma(\zeta_{\mathfrak{p}}))) = K(D_n(\omega)(\zeta_{\mathfrak{p}})), \quad \sigma \in \text{Gal}(\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K_{\mathfrak{p}^{n+1}}).$$

The induction hypothesis also says that $v_{0,n}(D_1(b_d)(\zeta_{\mathfrak{p}})D_{n-1}(\omega)(\zeta_{\mathfrak{p}})) = q^d$ for all $j = 2, \dots, n$ and $v_{0,n}(D_{n-j}(\omega)(\zeta_{\mathfrak{p}})) \geq q^{1+d}$. In particular, the element $\xi_n \in \mathbb{F}_{q^d}K_{\mathfrak{p}^n}$ defined in (8) with $\zeta = \zeta_{\mathfrak{p}}$ satisfies $v_{0,n}(\xi_n) = q^d$. In a similar way, one proves that $v_{d-k,n}(\xi_n) = q^{d+k}$, for $k = 1, \dots, d-1$. This gives (23) and it remains to apply Lemma 3.1 to conclude the proof. \square

Remark 3.4 The proof of Theorem 3.3 goes in the same direction of [15, Theorem IV], which corresponds to the case $n = 0$.

Proof of Theorem 1.1. By Hayes [9, Theorem 7.1], the field E is the compositum of \mathbb{F}_q^{ac} and the fields $K_{\mathfrak{p}^{n+1}}$ for $n \geq 0$, where \mathfrak{p} runs through the primes of A . Theorem 1.1 then follows from Theorem 3.3. \square

Proof of Corollary 1.2. In view of Theorem 1.1, it suffices to show that if \mathfrak{p} is a prime of A of degree d and if $\zeta_{\mathfrak{p}}$ is a root of \mathfrak{p} , then, for $n \geq 0$,

$$K(\tilde{\tau}^{-1}D_n(\mathfrak{L})(\zeta_{\mathfrak{p}})) = \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}. \quad (24)$$

By (4) and by the fact that D_n and τ commute, we observe that $\nu_n := D_n((t - \theta)\omega)(\zeta_{\mathfrak{p}}) = D_n(\omega)(\zeta_{\mathfrak{p}}^{q^{d-1}})^q$. By Theorem 3.3, $K(\nu_n) \subset \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$. If $K(\nu_n) \neq \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$, the extension $\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K(\nu_n)$ would be purely inseparable. But this is impossible since $\mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}/K$ is a separable extension. Hence, we have that $K(\nu_n) = \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$. By Leibniz rule and by induction on $n \geq 0$, we obtain:

$$K\left(D_n\left(\frac{1}{(t - \theta)\omega}\right)(\zeta)\right) = \mathbb{F}_{q^d}K_{\mathfrak{p}^{n+1}}$$

and (24) follows from (3). Now, Corollary 1.2 follows from yet another application of [9, Theorem 7.1]. \square

3.1 Compatibility with class field theory

We consider here the subring $\mathbb{B}_{d,\infty} \subset \mathbb{F}_{q^d}K_{\infty}(\tilde{\tau})[[t]]$ whose elements are the formal series which converge in $D^\circ = \{z \in \mathbb{C}_\infty; |z| < 1\}$. Let \mathfrak{p} be a prime of degree d and $\zeta = \zeta_{\mathfrak{p}}$ one of its roots. We have, in particular, that $\omega_\zeta \in \mathbb{B}_{d,\infty}$. We endow $\mathbb{B}_{d,\infty}$ with the $\mathbb{F}_{q^d}[t]$ -linear extension of the endomorphism $\tilde{\tau}$ defined in (13) (we keep the same notation).

Lemma 3.5 *The subset of $\mathbb{B}_{d,\infty}$ whose elements f are such that*

$$\tilde{\tau}(f) = (t + \zeta - \theta)f$$

is the free $\mathbb{F}_{q^d}[[t]]$ -module generated by ω_ζ .

Proof. This is clear by (14) and by the fact that the constant subring $\mathbb{B}_{d,\infty}^{\tilde{\tau}=1} = \{f \in \mathbb{B}_{d,\infty}; \tilde{\tau}(f) = f\}$ is equal to $\mathbb{F}_{q^d}[[t]]$. \square

Let K^{ab} be the maximal abelian extension of K in \mathbb{C}_∞ . Let σ be in $\text{Gal}(K^{ab}/\mathbb{F}_{q^d}K)$. The actions of σ and $\tilde{\tau}$ over K^{ab} commute. If we extend the action of σ $\mathbb{F}_{q^d}[[t]]$ -linearly to $K^{ab}[[t]]$, we obtain, thanks to Lemma 3.5, a representation

$$\rho_\zeta : \text{Gal}(K^{ab}/\mathbb{F}_{q^d}K) \rightarrow \text{GL}_1(\mathbb{F}_{q^d}[[t]]). \quad (25)$$

We consider now the idèle group $I_K = K^\times \times \prod_{\mathfrak{p}}' \widehat{A}_{\mathfrak{p}}^\times \times K_\infty^\times$, where $\widehat{A}_{\mathfrak{p}}$ denotes the completion of A at the place \mathfrak{p} and the product is the restricted one as usual (we denote by $\widehat{A}_{\mathfrak{p}}^\times$, K_∞^\times etc. the groups of invertible elements of the corresponding rings). We denote by

$$(\cdot, K^{ab}/K) : I_K \rightarrow \text{Gal}(K^{ab}/K)$$

the global norm residue symbol. In the next proposition we compute $(x, K^{ab}/K)(\omega_\zeta)$ for $x \in I_K$. We recall that $v_\infty : K_\infty^\times \rightarrow \mathbb{Z}$ is the ∞ -adic valuation normalized by $v_\infty(\theta) = -1$.

Proposition 3.6 *Let \mathfrak{p} be a prime of A of degree d and let ζ be one of its roots. Let x be an element of I_K . The following properties hold:*

1. *If $x = (\dots, 1, x_\infty, 1, \dots)$ with $x_\infty \in K_\infty^\times$, then*

$$(x, K^{ab}/K)(\omega_\zeta) = \text{sgn}(x) \omega_{\zeta^{q^{v_\infty(x_\infty)}}}.$$

2. *If $x = (\dots, 1, x_{\mathfrak{p}'}, 1, \dots)$ with $x_{\mathfrak{p}'} \in \widehat{A}_{\mathfrak{p}'}^\times$ and $\mathfrak{p} \neq \mathfrak{p}'$, then $(x, K^{ab}/K)(\omega_\zeta) = \omega_\zeta$.*

3. *If $x = (\dots, 1, x_{\mathfrak{p}}, 1, \dots)$ with $x_{\mathfrak{p}} \in \widehat{A}_{\mathfrak{p}}^\times$, then*

$$(x, K^{ab}/K)(\omega_\zeta) = \rho_\zeta((x, K^{ab}/K))(\omega_\zeta) = \psi(x_{\mathfrak{p}})^{-1} \omega_\zeta,$$

where $\psi_\zeta : \widehat{A}_{\mathfrak{p}} \rightarrow \mathbb{F}_{q^d}[[t]]$ is the unique continuous homomorphism of \mathbb{F}_{q^d} -algebras determined by $\psi_\zeta(\theta - \zeta) = t$.

Proof. 1. This follows from the fact that for all $n \geq 0$, any completion at a place above ∞ of $\mathbb{F}_{q^d} K_{P^{n+1}}$ is K_∞ -isomorphic to $\mathbb{F}_{q^d} K_\infty(\tilde{\pi})$.

2. This is clear.

3. Let ι_ζ be the embedding $\mathbb{F}_{q^d}[\theta] \rightarrow \widehat{A}_{\mathfrak{p}}$ such that $\iota_\zeta(\zeta - \theta) \in \mathfrak{p}\widehat{A}_{\mathfrak{p}}$. Identifying $\mathbb{F}_{q^d}A$ with its image via ι_ζ , we have $\widehat{A}_{\mathfrak{p}} = \mathbb{F}_{q^d}[[\theta - \zeta]]$. Let $u \in \widehat{A}_{\mathfrak{p}}^\times$ and let $a \in A$, $a \equiv u^{-1} \pmod{\mathfrak{p}^{n+1}}$, then, by [11, Theorem 5.5]:

$$(u, K^{ab}/K)(\lambda_{\mathfrak{p}^{n+1}}) = C_a(\lambda_{\mathfrak{p}^{n+1}}).$$

By (17) and induction:

$$(u, K^{ab}/K)(D_n(\omega)(\zeta)) = \widetilde{C}_a(D_n(\omega)(\zeta)).$$

This implies, by (16), that $\rho_\zeta((u, K^{ab}/K)) = \psi_\zeta(u)^{-1}$ for all $u \in \widehat{A}_{\mathfrak{p}}^\times$. □

4 Final remark

The element $f = 1 \otimes \theta - \theta \otimes 1 = t - \theta \in \mathbb{T}$ is the *shtuka* function associated to the Carlitz module C (see [8, Example 7.11.8]). We have observed in (9) that $\omega = -\exp_C(\tilde{\pi}f^{-1}) \in \mathbb{T}$ (note that $f \in \mathbb{T}^\times$).

For $n \geq 0$, we denote by $A^+(n)$ the set of monic polynomials a of A such that $\deg_\theta(a) = n$. By a variant of Anderson *log-algebraic theorem* ([12, §4], see also [6, §8]):

$$\sum_{n \geq 0} \sum_{a \in A^+(n)} a^{-1} C_a(X) = \log_C(X),$$

with X a variable in \mathbb{T} , and where $\log_C(X)$ denotes the Carlitz logarithm, defined by $\|X\| < q^{\frac{q}{q-1}}$ (the local inverse function at 0 of \exp_C). This implies that $\log_C(\omega) = \mathfrak{L}\omega$ yielding (3).

The shtuka function f specializes over \mathbb{F}_q^{ac} to Jacobi-Thakur's sums (see [16] Theorem 1.2), and as shown in the present paper (Theorem 2.9), ω specializes over \mathbb{F}_q^{ac} to Gauss-Thakur sums.

By a suitable variant of Anderson's log-algebraic Theorem for sign normalized rank one Drinfeld modules (Anderson, [1, Theorem 5.1.1] and [2, Theorem 3]) and with the use of the shtuka function associated to such a Drinfeld module (connected to Jacobi sums thanks to Thakur's [16, Theorem 1.2]) it is conceivable to generalize (3), Theorem 1.1 and Corollary 1.2 to the more general setting of $A = \Gamma(\mathcal{X} \setminus \{\infty\}, \mathcal{O}_{\mathcal{X}})$, where \mathcal{X}/\mathbb{F}_q is a smooth projective curve (geometrically irreducible) and ∞ a chosen \mathbb{F}_q -rational point. A . We hope to come back to this question in the near future.

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