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Abstract

Let $\mathfrak{g}$ be a complex simple Lie algebra with a Borel subalgebra $\mathfrak{b}$. Consider the semidirect product $I\mathfrak{b} = \mathfrak{b} \ltimes \mathfrak{b}^*$, where the dual $\mathfrak{b}^*$ of $\mathfrak{b}$ is equipped with the coadjoint action of $\mathfrak{b}$ and is considered as an abelian ideal of $I\mathfrak{b}$. We describe the automorphism group $\mathrm{Aut}(I\mathfrak{b})$ of the Lie algebra $I\mathfrak{b}$. In particular we prove that it contains the automorphism group of the extended Dynkin diagram of $\mathfrak{g}$. In type A_n, the dihedral subgroup was recently proved to be contained in $\mathrm{Aut}(I\mathfrak{b})$ by Dror Bar-Natan and Roland van der Veen in [1] (where $I\mathfrak{b}$ is denoted by $I\mathfrak{u}_n$). Their construction is ad hoc and they asked for an explanation which is provided by this note. Let $\mathfrak{n}$ denote the nilpotent radical of $\mathfrak{b}$. We obtain similar results for $\overline{I\mathfrak{b}} = \mathfrak{b} \ltimes \mathfrak{n}^*$ that is both an Inönü-Wigner contraction of $\mathfrak{g}$ and the quotient of $I\mathfrak{b}$ by its center.

1. Introduction

Given any complex Lie algebra $\mathfrak{a}$, one can consider the semi-direct product $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$. where $\mathfrak{a}^*$ is the dual of $\mathfrak{a}$, considered as an abelian ideal, and $\mathfrak{a}$ acts on $\mathfrak{a}^*$ via the coadjoint action. The pair $(I\mathfrak{a}, \mathfrak{a})$ is an example of the Drinfeld double construction with zero co-bracket.

As mentioned in [1], for applications in knot theory and representation theory, the most important case is when $\mathfrak{a} = \mathfrak{b}$ is the Borel subalgebra of some simple Lie algebra $\mathfrak{g}$. It is precisely the situation studied here. In addition to [1], several examples of these algebras appear with variations in the literature. In [8], Nappi-Wittney use the case when $\mathfrak{g} = \mathfrak{sl}_2$ in conformal field theory. Several authors also consider $\overline{I\mathfrak{b}} := \mathfrak{b} \ltimes \mathfrak{n}^*$ where $\mathfrak{n}$ is the derived subalgebra of $\mathfrak{b}$. It is the quotient of $I\mathfrak{b}$ by its center. Note that $\mathfrak{b} \ltimes \mathfrak{n}^*$ is a contraction of $\mathfrak{g}$ (see Section 2.1 for details). When $\mathfrak{g} = \mathfrak{gl}_n$, this algebra appears in an associative setting in Knutson and Zinn-Justin's work [6], see below. In [4, 3], Feigin uses $\mathfrak{b} \ltimes \mathfrak{n}^*$ in order to study degenerate flag varieties for $\mathfrak{g} = \mathfrak{sl}_n$. For a general semisimple Lie algebra $\mathfrak{g}$, in [9], Panyushev and Yakimova study the invariants of $\mathfrak{b} \ltimes \mathfrak{n}^*$ under the action of their adjoint group. Finally, in [10, 11], similar considerations are studied replacing $\mathfrak{b}$ by an arbitrary parabolic subalgebra of $\mathfrak{g}$.

The aim of this note is to give new interpretations of $I\mathfrak{b}$ and $\overline{I\mathfrak{b}}$ in the language of Kac-Moody algebras and to completely describe the automorphism groups of $I\mathfrak{b}$ and $\overline{I\mathfrak{b}}$.

Before describing this group, we introduce some notation. Let r denote the rank of $\mathfrak{g}$ and G the adjoint group with Lie algebra $\mathfrak{g}$. Let B be the Borel subgroup of G with $\mathfrak{b}$ as

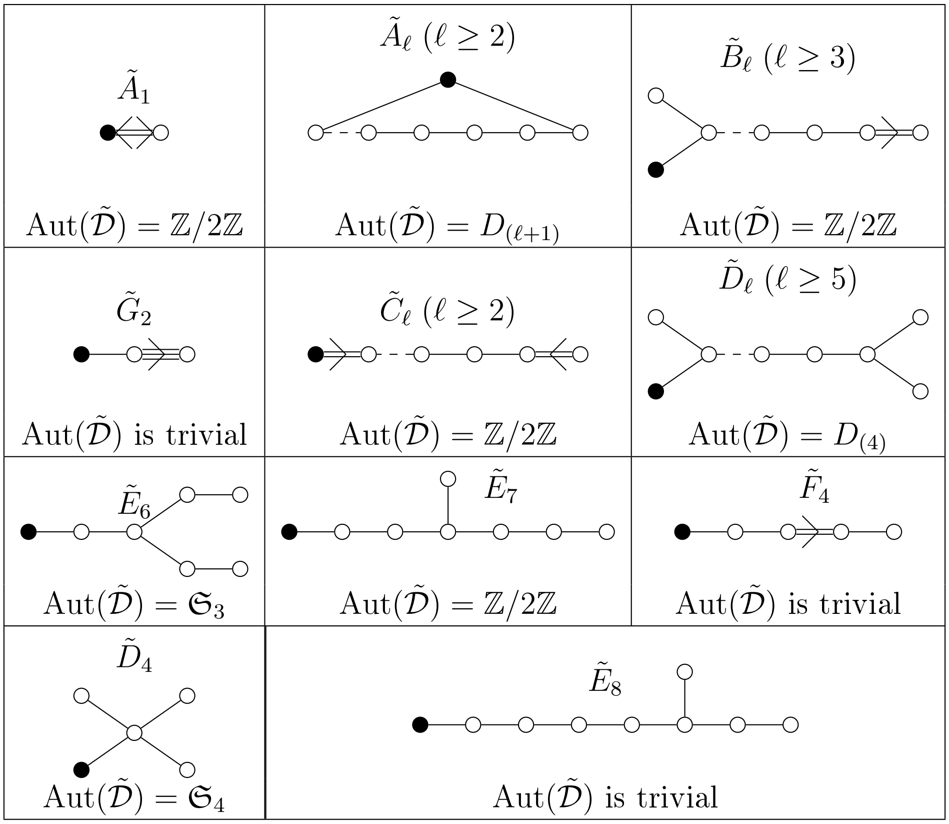

Figure 1: Extended Dynkin diagrams and their automorphisms

Lie algebra. Consider two abelian additive groups: the quotient $\mathfrak{g}/\mathfrak{b}$ and the space $\mathcal{M}_r(\mathbb{C})$ of $r \times r$-matrices.

An important ingredient is the extended Dynkin diagram of $\mathfrak{g}$. On Figure 1, these diagrams and their automorphism groups are shortly recalled (see Section 2.2). The notation $D_{(\ell)}$ stands for the dihedral group of order 2ℓ, not to be confused with the Dynkin diagram of type D_ℓ.

The following is the main result of the paper (see also Theorem 16 below).

Theorem 1. *The neutral component* $\mathrm{Aut}(I\mathfrak{b})^\circ$ *of the automorphism group* $\mathrm{Aut}(I\mathfrak{b})$ *of the Lie algebra* $I\mathfrak{b}$ *decomposes as*

$$\mathbb{C}^* \ltimes \Big((B \ltimes \mathfrak{g}/\mathfrak{b}) \times \mathcal{M}_r(\mathbb{C}) \Big).$$

The group of components $\mathrm{Aut}(I\mathfrak{b})/\mathrm{Aut}(I\mathfrak{b})^\circ$ *is isomorphic to the automorphism group of the extended Dynkin diagram of* $\mathfrak{g}$ *and can be lifted to a subgroup of* $\mathrm{Aut}(I\mathfrak{b})$.

The details of how these subgroups act on $I\mathfrak{b}$ are given in Section 3. Section 4 explains how the semidirect products are formed.

One of the amazing facts is that the extended Dynkin diagram of $\mathfrak{g}$ plays a crucial role in $\mathrm{Aut}(I\mathfrak{b})$. On one hand, we explain this by constructing the extended Cartan matrix of $\mathfrak{g}$ in terms of $I\mathfrak{b}$ in Section 3.1. On the other hand, this diagram is the Dynkin diagram of the untwisted affine Lie algebra constructed from the loop algebra of $\mathfrak{g}$. A second explanation is given by Theorem 4 that realizes $I\mathfrak{b}$ as a subquotient of the affine Lie algebra associated to $\mathfrak{g}$.

More generally, $I\mathfrak{b}$ is a degeneration $\lim_{\epsilon\to 0}\mathfrak{g}_+^\epsilon$ with $\mathfrak{g}_+^\epsilon \cong \mathfrak{g}\oplus\mathfrak{h}$ for $\epsilon\in\mathbb{C}\setminus\{0\}$. In Section 2, we explain how to interpret this degeneration in the affine Lie algebra setting. We also study the possible lifting of $\theta\in\mathrm{Aut}(\tilde{\mathcal{D}})$ to $\mathrm{Aut}(\mathfrak{g}_+^\epsilon)$, see Section 3.5.

Link with other works. In [6], Knutson and Zinn-Justin defined a degeneration $\bullet$ of the standard associative product on $\mathcal{M}_n(\mathbb{C})$. Let $\mathfrak{b}$ denote the set of upper triangular matrices. Identifying the vector space $\mathcal{M}_n(\mathbb{C})$ with $\mathfrak{b}\times\mathcal{M}_n(\mathbb{C})/\mathfrak{b}$ in a natural way one gets

$$(R,L)\bullet(V,M)=(RV,RM+LV),$$

for any $R,V\in\mathfrak{b}$ and $L,M\in\mathcal{M}_n(\mathbb{C})/\mathfrak{b}$. The Lie algebra of the group $(\mathcal{M}_n(\mathbb{C}),\bullet)^\times$ of invertible elements of this algebra is $\mathfrak{b}\ltimes\mathcal{M}_n(\mathbb{C})/\mathfrak{b}$, where the product is defined similarly to that of $I\mathfrak{b}$. Note also that a cyclic automorphism appears in [6]. It corresponds to the "unexpected cyclic automorphism" of [1] and, in our setting, to the cyclic automorphism of the extended Dynkin diagram of type A_{n-1}. Moreover [6, Proposition 2], which realizes $(\mathcal{M}_n(\mathbb{C}),\bullet)$ as a subquotient of $\mathcal{M}_n(\mathbb{C}[t])$, is similar to our Theorem 4.

A generalization of $\overline{I\mathfrak{b}}$ is the following: fix a simple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Let $\mathfrak{n}_\mathfrak{p}^-(\cong\mathfrak{g}/\mathfrak{p})$ be the nilradical of a parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. Then $\mathfrak{q}_\mathfrak{p}:=\mathfrak{p}\ltimes\mathfrak{n}_\mathfrak{p}^-$ is also a degeneration of $\mathfrak{g}$. In the study of semi-invariants of $\mathfrak{q}_\mathfrak{p}$ some data linked with the extended Dynkin diagram also come up in [13, Theorem 5.5] (Borel case) and in [11, Proposition 5.2.1] (general case). In type A_{n-1}, standard parabolics are characterized by an ordered partition $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_k)$ of n. Transforming $\boldsymbol{\lambda}$ into $\boldsymbol{\mu}:=(\lambda_k,\lambda_1,\dots,\lambda_{k-1})$, the cyclic action of $\mathbb{Z}/n\mathbb{Z}$ coming from the symmetries of the extended Dynkin diagrams described in [1] allows to write $\mathfrak{q}_{\mathfrak{p}_{\boldsymbol{\lambda}}}\cong\mathfrak{q}_{\mathfrak{p}_{\boldsymbol{\mu}}}$. This explains many symmetries noted in [11], see (3.9) in *loc. cit.*.

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2. The Lie algebras $I\mathfrak{b}$, $\mathfrak{g}_+^\epsilon$ and $\mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]$

2.1. Definitions of $I\mathfrak{b}$ and $\mathfrak{g}_+^\epsilon$

Let $\mathfrak{g}$ be a complex simple Lie algebra with Lie bracket denoted by $[\ ,\ ]$. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{b}$. Let $\mathfrak{b}^-$ be the Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ which is opposite to $\mathfrak{b}$. Set $\mathcal{V}=\mathfrak{b}\oplus\mathfrak{b}^-$ viewed as a vector space. In this section, we define the Lie bracket $[\ ,\ ]_\epsilon$ on $\mathcal{V}$ depending on the complex parameter ϵ, interpolating between $I\mathfrak{b}$ and the direct product $\mathfrak{g}\oplus\mathfrak{h}$.

Let $\mathfrak{n}$ and $\mathfrak{n}^-$ denote the derived subalgebras of $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively. Fix $\epsilon \in \mathbb{C}$. Define the skew-symmetric bilinear bracket $[\ ,\ ]_\epsilon$ on $\mathcal{V}$ by

$$
\begin{array}{lll}
[x, x']_\epsilon = [x, x'] & \forall x, x' \in \mathfrak{b} & \\
[y, y']_\epsilon = \epsilon[y, y'] & \forall y, y' \in \mathfrak{b}^- & \\
[x, y]_\epsilon = (\epsilon X + \epsilon\frac{H}{2}, \frac{H}{2} + Y) & \forall x \in \mathfrak{b}\ y \in \mathfrak{b}^- & \text{where } [x, y] = X + H + Y \in \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-
\end{array}
$$

Then $[\ ,\ ]_\epsilon$ satisfies the Jacobi identity (see discussion after (3) for a proof). Endowed with this Lie bracket, $\mathcal{V}$ is denoted by $\mathfrak{g}_+^\epsilon$. The linear map

$$
\begin{array}{rccc}
\varphi_\epsilon : & \mathfrak{b} \oplus \mathfrak{b}^- & \longrightarrow & \mathfrak{b} \oplus \mathfrak{b}^- \\
& (x, y) & \longmapsto & (x, \epsilon y) \qquad \text{for any } x \in \mathfrak{b},\ y \in \mathfrak{b}^-
\end{array}
$$

allows to interpret $\mathfrak{g}_+^\epsilon$ as an Inönü-Wigner contraction [5] of $\mathfrak{g}_+^1$. Indeed, for any nonzero ϵ, we have

$$
[X, Y]_\epsilon = \varphi_\epsilon{}^{-1}([\varphi_\epsilon(X), \varphi_\epsilon(Y)]_1) \qquad \forall X, Y \in \mathcal{V}. \tag{1}
$$

We now describe $\mathfrak{g}_+^1$. Using the triangular decomposition

$$
\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \tag{2}
$$

one defines the injective linear map

$$
\begin{array}{rccc}
\iota_\mathfrak{g}^1 : & \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- & \longrightarrow & \mathfrak{g}_+^1 \\
& (\xi, \alpha, \zeta) & \longmapsto & (\xi + \frac{\alpha}{2}, \frac{\alpha}{2} + \zeta)
\end{array}
$$

and checks that it is a Lie algebra homomorphism whose image is an ideal of $\mathfrak{g}_+^1$. Moreover, the image of

$$
\begin{array}{rccc}
\iota_\mathfrak{h}^1 : & \mathfrak{h} & \longrightarrow & \mathfrak{g}_+^1 \\
& \alpha & \longmapsto & (-\alpha, \alpha)
\end{array}
$$

is the center of $\mathfrak{g}_+^1$ and, as Lie algebras,

$$
\mathfrak{g}_+^1 = \iota_\mathfrak{g}^1(\mathfrak{g}) \oplus \iota_\mathfrak{h}^1(\mathfrak{h}). \tag{3}
$$

Observe that we never used the Jacobi identity for $[\ ,\ ]_1$ to prove the isomorphism (3). Hence, we can deduce from it that $[\ ,\ ]_1$ satisfies the Jacobi identity. Then, the expression (5) implies that $[\ ,\ ]_\epsilon$ satisfies the Jacobi identity for any nonzero ϵ. Since this property is closed on the space of bilinear maps, it is satisfied by $[\ ,\ ]_0$ too.

Let $\mathfrak{g}$ be a complex simple Lie algebra with Lie bracket denoted by $[\ ,\ ]$. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $\mathfrak{b}^-$ be the Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ which is opposite to $\mathfrak{b}$. Set $\mathcal{V} = \mathfrak{b} \oplus \mathfrak{b}^-$ viewed as a vector space. In this section, we define the Lie bracket $[\ ,\ ]_\epsilon$ on $\mathcal{V}$ depending on the complex parameter ϵ, interpolating between $I\mathfrak{b}$ and the direct product $\mathfrak{g} \oplus \mathfrak{h}$.

Let $\mathfrak{n}$ and $\mathfrak{n}^-$ denote the derived subalgebras of $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively, so that we can consider the triangular decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

The following is then an isomorphism of vector spaces

$$\begin{array}{rccl} \iota^1 : & \mathfrak{g} \oplus \mathfrak{h} \cong (\mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-) \oplus \mathfrak{h} & \longrightarrow & \mathcal{V} = \mathfrak{b} \oplus \mathfrak{b}^- \\ & ((\xi, \alpha, \zeta), \alpha') & \longmapsto & (\xi + \frac{\alpha}{2} - \alpha', \frac{\alpha}{2} + \zeta + \alpha') \end{array} \tag{4}$$

and the Lie bracket on $\mathfrak{g} \oplus \mathfrak{h}$ induces a bracket $[\ ,\ ]_1$ on $\mathcal{V} := \mathfrak{b} \oplus \mathfrak{b}^-$.

Then, for $\epsilon \in \mathbb{C} \setminus \{0\}$, we consider the isomorphism of vector space

$$\begin{array}{rccl} \varphi_\epsilon : & \mathfrak{b} \oplus \mathfrak{b}^- & \longrightarrow & \mathfrak{b} \oplus \mathfrak{b}^- \\ & (x, y) & \longmapsto & (x, \epsilon y) \end{array}$$

which allows us to define a modified Lie bracket $[\ ,\ ]_\epsilon$ on $\mathcal{V}$ via

$$[X, Y]_\epsilon = {\varphi_\epsilon}^{-1}([\varphi_\epsilon(X), \varphi_\epsilon(Y)]_1) \qquad \forall X, Y \in \mathcal{V}. \tag{5}$$

More explicitly, this yields

$$\begin{array}{lll} [x, x']_\epsilon = [x, x'] & \forall x, x' \in \mathfrak{b} & \\ [y, y']_\epsilon = \epsilon[y, y'] & \forall y, y' \in \mathfrak{b}^- & \\ [x, y]_\epsilon = (\epsilon X + \epsilon\frac{H}{2}, \frac{H}{2} + Y) & \forall x \in \mathfrak{b}\ y \in \mathfrak{b}^- & \text{where } [x, y] = X + H + Y \in \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-. \end{array}$$

This formula is also defined for $\epsilon = 0$, thus defining a bilinear map $[\ ,\ ]_0$. Since the property of being a Lie bracket is closed among the space of bilinear maps, we see that $[\ ,\ ]_0$ is also a Lie bracket.

For $\epsilon \in \mathbb{C}$, we define the Lie algebra $\mathfrak{g}_+^\epsilon$ as $\mathcal{V}$ endowed with $[\ ,\ ]_\epsilon$. By construction we have $\mathfrak{g}_+^\epsilon \cong \mathfrak{g} \oplus \mathfrak{h}$ when $\epsilon \neq 0$.

Consider now $I\mathfrak{b}$ with its Lie bracket $[\ ,\ ]_{I\mathfrak{b}}$ defined as follows: $\mathfrak{b}^*$ is an abelian ideal on which $\mathfrak{b}$ acts by the coadjoint action. Denote by $\kappa : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ the Killing form on $\mathfrak{g}$. Since the orthogonal complement of $\mathfrak{b}$ with respect to κ is $\mathfrak{n}$, $\mathfrak{b}^*$ identifies with $\mathfrak{g}/\mathfrak{n}$ as a $\mathfrak{b}$-module. Identify $\mathfrak{g}/\mathfrak{n}$ with $\mathfrak{b}^-$ in a canonical way (that is by $y \in \mathfrak{b}^- \longmapsto y + \mathfrak{n}$) and denote by $\pi : \mathfrak{g} \longrightarrow \mathfrak{b}^-$ the quotient map. Then $I\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$ identifies with $\mathfrak{b} \oplus \mathfrak{b}^- = \mathcal{V}$. Let $[\ ,\ ]_I$ denote the Lie bracket transferred to $\mathcal{V}$ from $[\ ,\ ]_{I\mathfrak{b}}$. Let $x, x' \in \mathfrak{b}$ and $y, y' \in \mathfrak{b}^-$ and decompose $[x, y'] - [x', y]$ as $X + H + Y$ with respect to $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Then

$$[(x, y), (x', y')]_I = ([x, x'], H + Y). \tag{6}$$

We now describe $\mathfrak{g}_+^0$. The Lie bracket $[\ ,\ ]_0$ on $\mathcal{V} = \mathfrak{g}_+^0$ is given by

$$[(x, y), (x', y')]_0 = ([x, x'], \frac{H}{2} + Y). \tag{7}$$

Comparing (6) and (7), one gets that the following linear map η is a Lie algebra isomorphism between $\mathfrak{g}^0_+$ and $I\mathfrak{b}$:

$$\begin{array}{rcccc} \eta: & \mathcal{V} = \mathfrak{b} \oplus (\mathfrak{h} \oplus \mathfrak{n}^-) & \longrightarrow & \mathfrak{b} \oplus \mathfrak{b}^* = I\mathfrak{b} \\ & (x, h, y) & \longmapsto & (x, \kappa(2h + y, \square)). \end{array}$$

Replacing $\mathfrak{b}^-$ and $\mathfrak{b}^*$ by $\mathfrak{n}^-$ and $\mathfrak{n}^*$ respectively, one defines $\mathfrak{g}^\epsilon$ and one gets the isomorphsims $\mathfrak{g} \simeq \mathfrak{g}^\epsilon$ (for any $\epsilon \neq 0$) and $\mathfrak{g}^0 \simeq \overline{I\mathfrak{b}}$.

2.2. The affine Kac-Moody Lie algebra

The untwisted affine Kac-Moody Lie algebra $\mathfrak{g}^{\mathrm{KM}}$ is constructed from the simple Lie algebra $\mathfrak{g}$. We refer to [7, Chapters I and XIII] for the basic properties of $\mathfrak{g}^{\mathrm{KM}}$. Denote by $\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}})$ the one dimensional center of $\mathfrak{g}^{\mathrm{KM}}$. Consider the Borel subalgebra $\mathfrak{b}^{\mathrm{KM}}$ of $\mathfrak{g}^{\mathrm{KM}}$ and its derived subalgebra $\mathfrak{n}^{\mathrm{KM}}$. By killing the semi-direct product and the central extension from the construction of $\mathfrak{g}^{\mathrm{KM}}$, one gets

$$\begin{array}{rcl} \tilde{\mathfrak{g}} & := & [\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}]/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) \\ & \cong & \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}, \end{array}$$

and

$$\begin{array}{rcl} \tilde{\mathfrak{b}} & := & (\mathfrak{b}^{\mathrm{KM}} \cap [\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}])/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) \subset \tilde{\mathfrak{g}} \\ \tilde{\mathfrak{n}} & := & (\mathfrak{n}^{\mathrm{KM}} \cap [\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}])/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}]. \end{array}$$

Identify $\mathfrak{g}$ with the subspace $\mathbb{C} \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}}$. Note that $\mathfrak{g}^{\mathrm{KM}}/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) = \tilde{\mathfrak{g}} + \mathbb{C}d$ where d acts as the derivation $t\frac{d}{dt}$.

We consider the set of (positive) roots $\Phi^{(+)}$ (resp. $\tilde{\Phi}^{(+)}$) of $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathrm{KM}}$) and the set of simple roots Δ (resp. $\tilde{\Delta}$) with respect to $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ (resp. $\mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) \subset \mathfrak{b}^{\mathrm{KM}} \subset \mathfrak{g}^{\mathrm{KM}}$). We recall the following classical facts:

$$\mathfrak{n}^{KM} \cong \tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \tilde{\Phi}^+} \tilde{\mathfrak{g}}_\alpha$$

where $\tilde{\mathfrak{g}}_\alpha \cong \mathfrak{g}^{KM}_\alpha$ is the root space associated to α. Moreover, $\tilde{\mathfrak{n}}$ is generated, as a Lie algebra by the subspaces $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}}$. The identification of Δ with $\{\alpha \in \tilde{\Delta} \,|\, \alpha(d) = 0\}$ yields the above-described embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. Denoting by δ the indivisible positive imaginary root in $\tilde{\Phi}$, we have

$$\tilde{\Phi} = \{n\delta + \alpha \,|\, \alpha \in \Phi \cup \{0\}, n \in \mathbb{Z}\} \setminus \{0\}$$

$$\tilde{\Delta} = \Delta \cup \{\alpha_0 + \delta\}$$

where α_0 is the lowest root of Φ. Note that $\tilde{\mathfrak{g}}_{n\delta} = t^n \mathfrak{h}$ ($n \in \mathbb{Z}$), using the notation $\tilde{\mathfrak{g}}_0 := \mathfrak{h}$.

Finally, the extended Dynkin diagram can be reconstructed from the combinatorics of $\tilde{\Delta}$ in $\tilde{\Phi}$. Indeed, the nodes correspond to the elements of $\tilde{\Delta}$ and the non-diagonal entries $a_{\alpha,\beta}$ of the generalized Cartan matrix (encoding the arrows of the diagram) are $a_{\alpha,\beta} = -\max\{n \in \mathbb{N} | \beta + n\alpha \in \tilde{\Phi}\}$ by Serre relations.

We list in Figure 1 the extended Dynkin diagram $\tilde{\mathcal{D}}_{\mathfrak{g}}$ in each simple type. The black node corresponds to the simple root $\alpha_0+\delta$. We also provide the automorphism group of $\tilde{\mathcal{D}}_{\mathfrak{g}}$. Note that by the definition of $\mathfrak{g}^{\mathrm{KM}}$ given in [7, §1.1], any $\theta \in \mathrm{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ provides an automorphism $\theta^{KM} \in \mathrm{Aut}(\mathfrak{g}^{\mathrm{KM}})$ stabilizing both $\mathfrak{h}+\mathbb{C}d+\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}})$ and $\mathfrak{b}^{\mathrm{KM}}$ and permuting the generators e_α, f_α $(\alpha \in \tilde{\Delta})$ via $\theta^{KM}(e_\alpha) = e_{\theta(\alpha)}$ and $\theta^{KM}(f_\alpha) = f_{\theta(\alpha)}$. Since $\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}})$ and $[\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}]$ are characteristic in $\mathfrak{g}^{\mathrm{KM}}$, *i.e.* stabilized by any automorphism of Lie algebra, this yields an automorphism $\tilde{\theta} \in \mathrm{Aut}(\tilde{\mathfrak{g}})$. Note that some choices have to be made for $\theta^{KM}(d)$, but the automorphism $\tilde{\theta}$ only depends on the $\theta^{KM}(e_\alpha)$, $\theta^{KM}(f_\alpha)$ with $\alpha \in \tilde{\Delta}$, since those elements generate $\tilde{\mathfrak{g}}$.

It is unclear whether $\tilde{\theta}$ is $\mathbb{C}[t]$-linear in general. When it is $\mathbb{C}[t]$-linear, we mention some consequences in Remark 15. However, we can still get the following general result.

Lemma 2. *With the above notations, there exists* $\lambda \in \{\pm 1\}$ *such that*

$$\forall x \in \tilde{\mathfrak{g}},\ \tilde{\theta}(tx) = \lambda t \tilde{\theta}(x). \tag{8}$$

In particular, the automorphism $\tilde{\theta} \in \mathrm{Aut}(\tilde{\mathfrak{g}})$ *stabilizes* $t\tilde{\mathfrak{n}}$.
Moreover, $\lambda = 1$ *whenever the order of* θ *is odd.*

Proof. Note that, since θ^{KM} acts on the semi-group $\tilde{\Phi}^+$, it stabilizes the semi-group of positive imaginary roots $\mathbb{N}^*\delta$ and thus fixes its generator δ. In particular, in the additive group $\tilde{\Phi} \cup \{0\}$, we have $\theta^{KM}(\cdot + \delta) = \delta + \theta^{KM}(\cdot)$. Defining Ψ on $\tilde{\mathfrak{g}}$ via $\Psi(x) = \tilde{\theta}^{-1}(t^{-1}\tilde{\theta}(tx))$, we thus get that $\Psi_\alpha := \Psi_{|\tilde{\mathfrak{g}}_\alpha}$ is an invertible linear map on $\tilde{\mathfrak{g}}_\alpha$ for any $\alpha \in \tilde{\Phi} \cup \{0\}$. Since $\dim \tilde{\mathfrak{g}}_\alpha = 1$ for $\alpha \in \tilde{\Phi} \backslash \mathbb{Z}\delta$, we can thus define λ_α as the element of $\mathbb{C}^\times$ such that $\Psi_\alpha = \lambda_\alpha Id_{\tilde{\mathfrak{g}}_\alpha}$.

Let $\alpha, \beta \in \tilde{\Phi} \cup \{0\}$, $x_\alpha \in \tilde{\mathfrak{g}}_\alpha$, $x_\beta \in \tilde{\mathfrak{g}}_\beta$. By $\mathbb{C}[t]$-bilinearity of the bracket, we get

$$\Psi_{\alpha+\beta}([x_\alpha, x_\beta]) = \tilde{\theta}^{-1}(t^{-1}[\tilde{\theta}(tx_\alpha), \tilde{\theta}(x_\beta)]) = [\Psi_\alpha(x_\alpha), x_\beta]. \tag{9}$$

For $\alpha = 0$, $x_\alpha = h \in \mathfrak{h}$ and $\beta \in \tilde{\Phi} \setminus \mathbb{Z}\delta$, we get

$$\lambda_\beta \beta(h) x_\beta = \Psi_\beta(\beta(h)x_\beta) \overset{(9)}{=} \beta(\Psi_0(h))x_\beta. \tag{10}$$

In particular, Ψ_0 induces on $\mathfrak{h}^*$ a linear map ${}^t\Psi_0$ sending β to $\lambda_\beta\beta$ for each $\beta \in \Phi \subset \tilde{\Phi} \setminus \mathbb{Z}\delta$. If $\beta, \gamma \in \Delta$ correspond to connected nodes of the Dynkin diagram of $\mathfrak{g}$, then β, γ and $\beta+\gamma$ are eigenvectors of ${}^t\Psi_0$ so $\lambda_\beta = \lambda_\gamma$. By connectivity of the Dynkin diagram, we get that the λ_β $(\beta \in \Delta)$ are all equal to a single value λ. Since Δ generates $\mathfrak{h}^*$, we get $\Psi_0 = \lambda Id_{\tilde{\mathfrak{g}}_0}$.

For any $\beta \in \tilde{\Phi} \setminus \mathbb{Z}\delta$, we can choose $h \in \mathfrak{h}$ such that $\beta(h) \neq 0$. Applying (10) yields $\lambda_\beta \beta(h) x_\beta = \beta(\lambda h) x_\beta$, that is $\lambda_\beta = \lambda$.

When $\alpha = -\beta \in \Delta$, $n \in \mathbb{Z}$, we get $\Psi_{n\delta}(t^n[x_\alpha, x_{-\alpha}]) \overset{(9)}{=} [\Psi_\alpha(x_\alpha), t^n x_{-\alpha}] = \lambda t^n [x_\alpha, x_{-\alpha}]$. Since the $t^n[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_{-\alpha}]$ $(\alpha \in \Delta)$ generate $\tilde{\mathfrak{g}}_{n\delta}$, this yields $\Psi_{n\delta} = \lambda Id_{\tilde{\mathfrak{g}}_{n\delta}}$. Finally, we have proved that $\Psi = \lambda Id_{\tilde{\mathfrak{g}}}$ and this yields (8), with $\lambda \in \mathbb{C}$.

Let m be the order of θ. Equation (8) can be rewritten as $t^{-1}\tilde{\theta}t = \lambda\tilde{\theta}$ where $t^{\pm 1}$ denotes the multiplication by $t^{\pm 1}$ in $\tilde{\mathfrak{g}}$. This identity to the power m yields $\lambda^m = 1$.

In the setting of [7, Chapter XIII], the Cartan involution ω of $\tilde{\mathfrak{g}}$ sending each generator e_α ($\alpha \in \tilde{\Delta}$) to $-f_\alpha$ is given by

$$\omega(t^i x) = t^{-i}\mathring{\omega}(x) \qquad (i \in \mathbb{Z}, x \in \mathfrak{g})$$

where $\mathring{\omega}$ is the Cartan involution of $\mathfrak{g}$. As a consequence, $\omega t = t^{-1}\omega$. Also, $\omega \circ \tilde{\theta} \circ \omega(e_\alpha) = \omega \circ \tilde{\theta}(-f_\alpha) = -\omega(f_{\theta(\alpha)}) = e_{\theta(\alpha)} = \tilde{\theta}(e_\alpha)$ and the same computation gives $\omega \circ \tilde{\theta} \circ \omega(f_\alpha) = \tilde{\theta}(f_\alpha)$ so $\omega\tilde{\theta}\omega = \tilde{\theta}$. Then conjugating $t^{-1}\tilde{\theta}t = \lambda\tilde{\theta}$ by the involution ω yields $t\tilde{\theta}t^{-1} = \lambda\tilde{\theta}$. It follows from these equalities that $\lambda^2 = 1$. Hence $\lambda \in \{\pm 1\}$ with $\lambda = 1$ if m is odd.

Finally, $\tilde{\theta}$ permutes the generators of $\tilde{\mathfrak{n}}$: $(e_\alpha)_{\alpha\in\tilde{\Delta}}$. Hence $\tilde{\theta}$ stabilizes $\tilde{\mathfrak{n}}$ and $\tilde{\theta}(t\tilde{\mathfrak{n}}) = \pm t\tilde{\mathfrak{n}} = t\tilde{\mathfrak{n}}$

□

Remark 3. We also checked in several cases, including the cyclic automorphism in type A, that $\lambda = 1$. In such cases, $\tilde{\theta}$ then also stabilizes $(t-\epsilon)\tilde{\mathfrak{n}}$ for any $\epsilon \in \mathbb{C}$.

2.3. Realization of $\mathfrak{g}_+^\epsilon$

The Lie algebras $\tilde{\mathfrak{b}}$ and $\tilde{\mathfrak{n}}$ decompose as

$$\begin{aligned} \tilde{\mathfrak{b}} &= \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{n}^-, \\ \tilde{\mathfrak{n}} &= \mathbb{C}[t]\mathfrak{n} \oplus t\mathbb{C}[t]\mathfrak{b}^-. \end{aligned}$$

Moreover, $(t-\epsilon)\tilde{\mathfrak{n}}$ is an ideal of $\tilde{\mathfrak{b}}$, and $\tilde{\mathfrak{b}}/((t-\epsilon)\tilde{\mathfrak{n}})$ is a Lie algebra.

Theorem 4. *Let $\epsilon \in \mathbb{C}$. The Lie algebras $\mathfrak{g}_+^\epsilon$ and $\tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}}$ are isomorphic. Similarly, $\mathfrak{g}^\epsilon$ is isomorphic to $\tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{b}}$.*

Proof. From Section 2.1, we have $\mathfrak{g}_+^1 = \mathfrak{b} \oplus \mathfrak{b}^-$ as vector spaces. Elements of $\mathfrak{g}_+^1$ will be written as couples with respect to this decomposition.

Let $\iota_{\mathfrak{g}}^1 : \mathfrak{g} \to \mathfrak{g}_+^1$ be $(\iota^1)_{|\mathfrak{g}\times\{0\}}$ where ι_1 is as in (4). Set $\widetilde{\mathfrak{g}_+^1} := \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}_+^1$ and extend $\iota_{\mathfrak{g}}^1$ to an injective $\mathbb{C}[t^{\pm 1}]$-linear map $\tilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}_+^1}$. Consider the subspace $\mathfrak{w} := \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$ that is a Lie subalgebra of $\widetilde{\mathfrak{g}_+^1}$. If $\epsilon \neq 0$, the Inönü-Wigner contraction (5) on $\mathfrak{g}_+^1$ with respect to the decomposition $\mathfrak{b} \oplus \mathfrak{b}^-$ gives rise to $\mathfrak{g}_+^\epsilon$ ($\epsilon \in \mathbb{C}$). We easily deduce that the linear map

$$\begin{array}{ccl} \mathfrak{g}_+^\epsilon & \longrightarrow & \mathfrak{w}/(t-\epsilon)\mathfrak{w} \\ (x,y) & \longmapsto & x + ty + (t-\epsilon)\mathfrak{w} \qquad \text{for any } x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-, \end{array} \tag{11}$$

is a Lie algebra isomorphism. For $\epsilon = 0$, it is still a linear isomorphism and, by continuity, a Lie algebra homomorphism.

Set $\mathfrak{b}_0^- := \iota_{\mathfrak{g}}^1(\mathfrak{b}^-) = \{(h,h) | h \in \mathfrak{h}\} \oplus \mathfrak{n}^-$. Observe that $t\mathfrak{b}_0^-$ is contained in $\mathfrak{w}$. Indeed, for any $h \in \mathfrak{h}$, the element $t(h,h) = t(h,0) + t(0,h)$ belongs to $\mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$. In particular, one gets a linear map induced by the inclusions of $\mathfrak{b}$ and $t\mathfrak{b}_0^-$ in $\mathfrak{w}$:

$$\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}.$$

One can easily check that it induces a linear isomorphism $\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}/(t-\epsilon)\mathfrak{w}$. Setting $\tilde{\mathfrak{b}}_{\mathfrak{w}} := \langle \mathfrak{b} \oplus t\mathfrak{b}_0^- \rangle_{Lie}$, the Lie subalgebra of $\mathfrak{w}$ generated by $\mathfrak{b} \oplus t\mathfrak{b}_0^-$, we thus get a Lie algebra isomorphism.

$$\tilde{\mathfrak{b}}_{\mathfrak{w}}/((t-\epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}) \longrightarrow \mathfrak{w}/(t-\epsilon)\mathfrak{w}. \tag{12}$$

Since, $\mathfrak{b} = \{(h,0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\mathfrak{n})$ and $\langle \iota_{\mathfrak{g}}^1(\mathfrak{n}) \oplus \iota_{\mathfrak{g}}^1(t\mathfrak{b}^-) \rangle_{Lie} = \iota_{\mathfrak{g}}^1(\langle \mathfrak{n} \oplus t\mathfrak{b}^- \rangle_{Lie}) = \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$, we have

$$\tilde{\mathfrak{b}}_{\mathfrak{w}} = \{(h,0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}}) \cong \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{b}}) \cong \tilde{\mathfrak{b}}, \tag{13}$$

the middle Lie algebra isomorphism being the identity on $\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ and sending $(h,0)$ to $\frac{1}{2}(h,h)$ for each $h \in \mathfrak{h}$. Moreover, $(t-\epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}} = (t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$. Indeed, $(t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ is contained in $(t-\epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}$, and $\mathfrak{b} \oplus t\mathfrak{b}_0^-$ is complementary to $(t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ in $\tilde{\mathfrak{b}}_{\mathfrak{w}}$.

We finally get the desired Lie isomorphism

$$\tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}} \overset{(13)}{\cong} \tilde{\mathfrak{b}}_{\mathfrak{w}}/(t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}}) \overset{(12)}{\cong} \mathfrak{w}/(t-\epsilon)\mathfrak{w} \overset{(11)}{\cong} \mathfrak{g}_+^{\epsilon}$$

□

In addition, we can make explicit the isomorphism of Theorem 4:

$$\begin{array}{rcccl} \gamma_\epsilon : & \mathfrak{g}_+^{\epsilon} & \overset{\cong}{\longrightarrow} & \tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}} & \\ & (x,0) & \longmapsto & x & \text{if } x \in \mathfrak{n} \\ & (0,y) & \longmapsto & ty & \text{if } y \in \mathfrak{n}^- \\ & (a,b) & \longmapsto & (a-\epsilon b) + 2tb & \text{if } a,b \in \mathfrak{h} \end{array}$$

and its inverse map is induced by

$$\begin{array}{rcccl} \theta : & \tilde{\mathfrak{b}} & \longrightarrow & \mathcal{V} & \\ & Px & \longmapsto & P(\epsilon)x & \text{if } x \in \mathfrak{n} \\ & tRy & \longmapsto & R(\epsilon)y & \text{if } y \in \mathfrak{n}^- \\ & Qh & \longmapsto & (\frac{Q(\epsilon)+Q(0)}{2}h, \frac{Q(\epsilon)-Q(0)}{2\epsilon}h) & \text{if } h \in \mathfrak{h}\ (\epsilon \neq 0) \\ & & & (Q(0)h, \frac{1}{2}Q'(0)h) & \text{if } h \in \mathfrak{h}\ (\epsilon = 0) \end{array}$$

Note that, in order to prove Theorem 4, we could alternatively have checked directly that θ is a surjective Lie algebra homomorphism from $\tilde{\mathfrak{b}}$ onto $\mathfrak{g}_+^{\epsilon}$ with kernel $(t-\epsilon)\tilde{\mathfrak{n}}$.

3. Some subgroups of Aut($I\mathfrak{b}$)

3.1. The roots of $I\mathfrak{b}$

From Sections 2.1 and 2.3, we can interpret the Lie algebra $I\mathfrak{b}$ in the Kac-Moody world via the isomorphism

$$\begin{array}{rcl} I\mathfrak{b} & \longrightarrow & \tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}} \\ (x,y) & \longmapsto & x+ty \end{array} \qquad \left(\begin{array}{c} x \in \mathfrak{b}, \\ y \in \mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n} \overset{\kappa}{\cong} \mathfrak{b}^* \end{array} \right)$$

From now on, this identification will be made systematically. In particular, we write $I\mathfrak{b} = \mathfrak{b} \oplus t\mathfrak{b}^-$. We first describe some basic properties of $I\mathfrak{b}$ in this language.

Lemma 5. 1. *The subalgebra* $\mathfrak{c} := \mathfrak{h} \oplus t\mathfrak{h}$ *is a Cartan subalgebra of* $I\mathfrak{b}$. *Namely,* $\mathfrak{c}$ *is abelian and equal to its normalizer.*

2. *Under the action of* $\mathfrak{c}$, $I\mathfrak{b}$ *decomposes as*

$$I\mathfrak{b} = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} t\mathfrak{g}_\alpha.$$

For $\alpha \in \Phi^+$, $\mathfrak{c}$ *acts on* $\mathfrak{g}_\alpha$ *with the weight* $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$. *For* $\alpha \in \Phi^-$, $\mathfrak{c}$ *acts on* $t\mathfrak{g}_\alpha$ *with the weight* $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$. *Here, we identified* $\mathfrak{c}^*$ *with* $\mathfrak{h}^* \times t\mathfrak{h}^*$ *in a natural way.*

3. *The set of* ad-*nilpotent elements of* $I\mathfrak{b}$ *is* $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}} = \mathfrak{n} \oplus t\mathfrak{b}^-$.

4. *The center of* $I\mathfrak{b}$ *is* $\mathfrak{z}(I\mathfrak{b}) = t\mathfrak{h}$.

5. *The derived subalgebra of* $I\mathfrak{b}$ *is* $[I\mathfrak{b}; I\mathfrak{b}] = \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$.

Proof. 1-2) The fact that $\mathfrak{c}$ is abelian and the decomposition in $\mathfrak{h}$-eigenspaces are clear from the definition of $\tilde{\mathfrak{g}}$. The action of $t\mathfrak{h}$ is zero since it sends $\tilde{\mathfrak{n}}$ to $t\tilde{\mathfrak{n}}$ that vanishes itself in $I\mathfrak{b}$. The decomposition of $I\mathfrak{b}$ in weight spaces under the action of $\mathfrak{c}$ follows. Then this decomposition also implies that $\mathfrak{c}$ is its own normalizer in $I\mathfrak{b}$.
3) The elements of $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ are clearly ad-nilpotent. From 2), an element with nonzero component in $\mathfrak{h}$ is not ad-nilpotent.
4) Since $t\mathfrak{h}$ acts as 0 on $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ and on $\mathfrak{h}$, we have $t\mathfrak{h} \subset \mathfrak{z}(I\mathfrak{b})$. The decomposition in weight spaces implies the converse inclusion.
5) The inclusion $[I\mathfrak{b}, I\mathfrak{b}] \subset \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is clear. On the other hand we deduce from the weight space decomposition that the subspaces $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \tilde{\Delta}}$ belong to $[I\mathfrak{b}, I\mathfrak{b}]$. Since they generate $\tilde{\mathfrak{n}}$ in $\tilde{\mathfrak{g}}$, the result follows. ☐

It follows from Lemma 5 and Theorem 4 that $\overline{I\mathfrak{b}} \cong I\mathfrak{b}/t\mathfrak{h} \cong \mathfrak{g}^0_+/\mathfrak{z}(\mathfrak{g}^0_+) \cong \mathfrak{g}^0$. Then it is straightforward from Lemma 5 and its proof that

- $\mathfrak{h}$ is a Cartan subalgebra of $\overline{I\mathfrak{b}}$.
- The non-zero $\mathfrak{h}$-weights (resp. weight spaces) on $\overline{I\mathfrak{b}}$ coincide with the non-zero $\mathfrak{c}$-weights (resp. weight space) on $I\mathfrak{b}$ via projection. In particular $\Phi(\overline{I\mathfrak{b}}) \cong \Phi(I\mathfrak{b}) \cong \Phi$.
- $[\overline{I\mathfrak{b}}, \overline{I\mathfrak{b}}] = \tilde{\mathfrak{n}}/t\tilde{\mathfrak{b}}$.

From Lemma 5 (2), the set $\Phi(I\mathfrak{b})$ of nonzero weights of $\mathfrak{c}$ acting on $I\mathfrak{b}$ identifies with Φ. It is also useful to embed $\Phi(I\mathfrak{b})$ in $\tilde{\Phi}$ by

$$\begin{array}{rccc} \varphi: & \Phi(I\mathfrak{b}) & \longrightarrow & \tilde{\Phi} \\ & \alpha \in \Phi^+ & \longmapsto & \alpha \\ & \alpha \in \Phi^- & \longmapsto & \delta + \alpha \end{array}$$

Indeed, the weight space $(I\mathfrak{b})_\alpha$ identifies with $\tilde{\mathfrak{g}}_{\varphi(\alpha)}$, for any $\alpha \in \Phi(I\mathfrak{b})$. In particular, for $\alpha, \beta \in \tilde{\Phi} \cup \{0\}$, we have $[I\mathfrak{b}_{\varphi^{-1}(\alpha)}, I\mathfrak{b}_{\varphi^{-1}(\beta)}] \subset I\mathfrak{b}_{\varphi^{-1}(\alpha+\beta)}$ with equality when $\alpha, \beta, \alpha + \beta \notin \{0, \delta\}$. Set also $\Delta(I\mathfrak{b}) = \varphi^{-1}(\tilde{\Delta}) = \Delta \cup \{\alpha_0\}$.

Lemma 6. 1. *The derived subalgebra of* $I\mathfrak{b}^{(1)} := [I\mathfrak{b}, I\mathfrak{b}]$ *is*

$$I\mathfrak{b}^{(2)} = t\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi(I\mathfrak{b})\setminus\Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha$$

2. *Assume that* $\mathfrak{g}$ *is not* $\mathfrak{sl}_2$*. For* $\alpha, \beta \in \Delta(I\mathfrak{b})$ $(\alpha \neq \beta)$*, the corresponding entry of the generalized Cartan Matrix of* $\mathfrak{g}^{\mathrm{KM}}$ *is given by*

$$a_{\alpha,\beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \Phi(I\mathfrak{b})\}.$$

Proof. 1) Recall that $\tilde{\mathfrak{n}}$ is generated as a Lie algebra by the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha\in\tilde{\Delta}}$. Thus, for weight reasons, the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha\in\tilde{\Phi}\setminus\tilde{\Delta}}$ are root spaces included in $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$. Since $\tilde{\Delta}$ is a linearly independent set, they are in fact the only root spaces not contained in $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$. Taking a quotient, this yields $\bigoplus_{\alpha\in\Phi(I\mathfrak{b})\setminus\Delta(I\mathfrak{b})}(I\mathfrak{b})_\alpha = I\mathfrak{b}^{(2)}$.

2) Recall that the statement is valid if we replace $\Phi(I\mathfrak{b})$ by $\tilde{\Phi}$, see Section 2.2. It is thus sufficient to show that

$$\beta + n\alpha \in \tilde{\Phi} \;\Rightarrow\; \beta + n\alpha \in \Phi(I\mathfrak{b}).$$

When $\alpha, \beta \in \Delta$, the statement is clear since $\Phi^+ \subset \Phi(I\mathfrak{b})$.
If $\beta = \delta+\alpha_0$, then $\beta+n\alpha \in \tilde{\Phi}$ means that $\alpha_0+n\alpha \in \Phi$. Expressing α_0 as a linear combination of simple roots, one gets only negative coefficients. Since $\mathfrak{g}$ is not $\mathfrak{sl}_2$, some of them remain negative in the expression of $\alpha_0 + n\alpha$, so this root has to lie in Φ^-. Thus $\beta + n\alpha \in \Phi(I\mathfrak{b})$.
If $\alpha = \delta + \alpha_0$, then $\beta + n\alpha \in \tilde{\Phi}$ means that $\beta + n\alpha_0 \in \Phi$. For height reasons, we must have $n \in \{0, 1\}$. Then, $\beta + n\alpha \in \Phi(I\mathfrak{b})$. □

Remark 7. One can observe that the first assertion of Lemma 6 is similar to

$$[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{\alpha\in\Phi^+\setminus\Delta} \mathfrak{b}_\alpha.$$

3.2. The adjoint subgroup of $\mathrm{Aut}(I\mathfrak{b})$

Let G be the adjoint group with Lie algebra $\mathfrak{g}$. Let T and B be the connected subgroups of G with Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$. Consider now $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$ equipped with the addition as an abelian algebraic group. The adjoint action of B on $\mathfrak{g}$ stabilizes $\mathfrak{n}$ and induces a linear action on $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$ by group isomorphisms. We can construct the semidirect product:

$$IB := B \ltimes \mathfrak{b}^-.$$

By construction the Lie algebra of IB identifies with $I\mathfrak{b}$. The adjoint action of IB on $I\mathfrak{b}$ is given by

$$\begin{array}{ccl} IB \times I\mathfrak{b} & \longrightarrow & I\mathfrak{b} \\ ((b, f), x + ty) & \longmapsto & b \cdot x + tb \cdot (y + [f, x] + \mathfrak{n}) \quad \text{for } b \in B,\ x \in \mathfrak{b} \text{ and } f, y \in \mathfrak{b}^-, \end{array} \tag{14}$$

where $y + [f, x] + \mathfrak{n}$ is viewed as an element of $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{b}^-$ and where $\cdot$ denotes the B-action on $\mathfrak{b}$ and on $\mathfrak{b}^-$. It induces a group homomorphism

$$\mathrm{Ad} : IB \longrightarrow \mathrm{Aut}(I\mathfrak{b})$$

with kernel $Z(IB) \cong (1, \mathfrak{h})$. In particular, one gets:

Lemma 8. *The image* $\mathrm{Ad}(IB)$ *is isomorphic to* $B \ltimes \mathfrak{g}/\mathfrak{b}$*.*

Note also that $\mathrm{Ad}(IB) = H \ltimes (N \ltimes \mathfrak{g}/\mathfrak{b})$ where N and H are the connected subgroups of B with respective Lie algebras $\mathfrak{n}$ and $\mathfrak{h}$. Since $\mathfrak{n} + t\mathfrak{b}^-$ is the set of ad-nilpotent elements of $I\mathfrak{b}$, we get the following result from (14).

Lemma 9.

1. *The group of elementary automorphisms*[1] $\mathrm{Aut}_e(I\mathfrak{b}) = \exp \mathrm{ad}(\mathfrak{n} + t\mathfrak{b}^-)$ *coincides with* $N \ltimes \mathfrak{g}/\mathfrak{b}$*.*
2. $\mathrm{Ad}(IB) = \exp \mathrm{ad}(I\mathfrak{b})$

3.3. A unipotent subgroup of $\mathrm{Aut}(I\mathfrak{b})$

Let $\mathfrak{a}$ be a Lie algebra. We consider the derived subalgebra $\mathfrak{a}^{(1)} := [\mathfrak{a}, \mathfrak{a}]$, the center $\mathfrak{z} := \mathfrak{z}(\mathfrak{a})$ and the quotient Lie algebra $\bar{\mathfrak{a}} := \mathfrak{a}/\mathfrak{z}$.

Any linear map $u \in \mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}, \mathfrak{z})$, defines a linear map $\bar{u} : \left\{ \begin{array}{ccc} \mathfrak{a} & \longrightarrow & \mathfrak{a} \\ X & \longmapsto & X + u(X + \mathfrak{a}^{(1)}) \end{array} \right.$.
Since u takes values in $\mathfrak{z}$ and vanishes on $\mathfrak{a}^{(1)}$, we have

$$[\bar{u}(X), \bar{u}(Y)] = [X + u(X), Y + u(Y)] = [X, Y] = [X, Y] + u([X, Y]) = \bar{u}([X, Y]).$$

In other words, $\bar{u}$ is a morphism of Lie algebras.

On the other hand, any $\theta \in \mathrm{Aut}(\mathfrak{a})$ stabilizes the center of $\mathfrak{a}$, and hence it induces an automorphism of $\bar{\mathfrak{a}}$. This yields a natural group homomorphism

$$R : \mathrm{Aut}(\mathfrak{a}) \to \mathrm{Aut}(\bar{\mathfrak{a}}). \tag{15}$$

Lemma 10. *Assume that* $\mathfrak{z}(\mathfrak{a}) \subset \mathfrak{a}^{(1)}$*. With the above notations, we have an exact sequence of groups*

$$\begin{array}{ccccc} 0 \longrightarrow \mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}, \mathfrak{z}) & \longrightarrow & \mathrm{Aut}(\mathfrak{a}) & \xrightarrow{R} & \mathrm{Aut}(\bar{\mathfrak{a}}) \\ u & \longmapsto & \bar{u} & & \end{array}$$

where $\mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}, \mathfrak{z})$ *is seen as the additive vector group.*

We denote

$$U := \{\bar{u} \mid u \in \mathrm{Hom}(I\mathfrak{b}/I\mathfrak{b}^{(1)}, \mathfrak{z}(I\mathfrak{b}))\}. \tag{16}$$

This lemma, together with Lemma 5, implies the following results

Corollary 11.

1. $(U, \circ)$ *is a normal subgroup of* $\mathrm{Aut}(I\mathfrak{b})$ *of dimension* $(\dim \mathfrak{h})^2$
2. $R(\mathrm{Aut}(I\mathfrak{b})) = \mathrm{Aut}(I\mathfrak{b})/U \subset \mathrm{Aut}(\overline{I\mathfrak{b}})$.

We will see in Lemma18 that the last inclusion is actually an equality (*i.e.* the sequence of Lemma 10 is a short exact sequence for $\mathfrak{a} = I\mathfrak{b}$)

[1] Recall that the *group of elementary automorphisms* of a Lie algebra $\mathfrak{a}$ is the group generated by the $\exp(\mathrm{ad}\, n)$ for $n \in \mathfrak{a}$ ad-nilpotent, cf. [12, 19.1.4]

Proof of Lemma 10. We have

$$(\bar{u}\circ\bar{v})(X) = (X+v(X)) + u(X+v(X)) = X + u(X) + v(X) = \overline{u+v}(X)$$

where the middle equality is due to $v(X)\in\mathfrak{z}\subset\mathfrak{a}^{(1)}\subset \mathrm{Ker}(u)$. So the map $u\mapsto\bar{u}$ is a semi-group homomorphism from $(\mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)},\mathfrak{z}),+)$ to $(\mathrm{End}(\mathfrak{a}),\circ)$. Since $(\mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^{(1)}],\mathfrak{z}),+)$ is actually a group, its image is contained in $\mathrm{Aut}(\mathfrak{a})$.

It is clear that the map $u\mapsto\bar{u}$ is injective and, since u takes values in $\mathfrak{z}$, that $R(\bar{u}) = Id_{\bar{\mathfrak{a}}}$. In order to prove exactness of the sequence at $\mathrm{Aut}(\mathfrak{a})$, there remains to prove the implication

$$\forall\theta\in\mathrm{Aut}(\mathfrak{a}),\ R(\theta) = Id_{\bar{\mathfrak{a}}}\Rightarrow ((\theta - Id)(\mathfrak{a})\subset\mathfrak{z}) \text{ and } ((\theta-Id)_{|\mathfrak{a}^{(1)}} = 0)$$

The first property is immediate. The second one follows from the fact that, for such a θ, we have $\theta([X,Y])\in[X+\mathfrak{z},Y+\mathfrak{z}] = [X,Y]$. □

3.4. The loop subgroup

Lemma 12. *The following map is an injective group homomorphism*

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathrm{Aut}(I\mathfrak{b}) \\ \tau & \longmapsto & \left(\begin{array}{rcll} \delta_\tau: I\mathfrak{b} & \longrightarrow & I\mathfrak{b} & \\ x & \longmapsto & x & \text{if } x\in\mathfrak{b} \\ ty & \longmapsto & \tau ty & \text{if } y\in\mathfrak{b}^- \end{array}\right). \end{array}$$

We denote by $D\subset\mathrm{Aut}(I\mathfrak{b})$ the image of this map.

Proof. It is a straightforward check on $\mathfrak{b}\ltimes t\mathfrak{b}^-$ that the δ_τ are automorphisms of $I\mathfrak{b}$. □

Remark 13. The map δ_τ corresponds to the change of variable $t\mapsto\tau t$ in the $\mathbb{C}[t]$-Lie algebra $\tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}}$. Moreover, the Lie algebra of D acts on $I\mathfrak{b}$ like $\mathbb{C}d$ where d is the derivation involved in the definition of $\mathfrak{g}^{KM}$.

3.5. Automorphisms stabilizing the Cartan subalgebra

For any $\alpha\in\Delta(I\mathfrak{b})$, fix generators e_α of $\tilde{\mathfrak{g}}$, $\alpha\in\tilde{\Delta}$ giving rise to elements $X_\alpha\in I\mathfrak{b}_\alpha$ in the corresponding root space $(I\mathfrak{b})_\alpha$. Set

$$\Gamma := \left\{\theta\in\mathrm{Aut}(I\mathfrak{b}) \;\middle|\; \begin{array}{l}\theta(\mathfrak{h})\subset\mathfrak{h} \\ \theta(\{X_\alpha : \alpha\in\Delta(I\mathfrak{b})\}) = \{X_\alpha : \alpha\in\Delta(I\mathfrak{b})\}\end{array}\right\}.$$

Note that, since $\mathfrak{c}$ is the sum of $\mathfrak{h}$ with $\mathfrak{z}(I\mathfrak{b})$ and since the center is characteristic, the elements of Γ also stabilize $\mathfrak{c}$.

Proposition 14. *The group Γ is isomorphic to the automorphism group of the affine Dynkin diagram of $\mathfrak{g}$.*

Proof. By construction, Γ induces an action on $\Delta(I\mathfrak{b})$. By Lemma 6 (2), we have for $g \in \Gamma$ and $\alpha, \beta \in \Delta(I\mathfrak{b})$:

$$\begin{aligned} a_{\alpha,\beta} &= -\max\{n | (\operatorname{ad} X_\alpha)^n (X_\beta) \neq 0\} \\ &= -\max\{n | g((\operatorname{ad} X_\alpha)^n (X_\beta)) \neq 0\} \\ &= -\max\{n | (\operatorname{ad} X_{g(\alpha)})^n (X_{g(\beta)}) \neq 0\} = a_{g(\alpha), g(\beta)}. \end{aligned}$$

Hence g actually induces an automorphism of the extended Dynkin diagram[2] and we thus obtain a group homomorphism

$$\Theta : \Gamma \to \operatorname{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}}).$$

We claim that Θ is surjective. Indeed, fix an automorphism θ of the group $\tilde{\mathcal{D}}_{\mathfrak{g}}$. As it was mentioned in Section 2.2, there exists $\tilde{\theta} \in \operatorname{Aut}(\tilde{\mathfrak{g}})$ which stabilizes both $\mathfrak{h}$ and $\tilde{\mathfrak{b}}$ and which permutes the generators $\{e_\alpha : \alpha \in \tilde{\Delta}\}$ and thus $\tilde{\Delta} \overset{\varphi}{\cong} \Delta(I\mathfrak{b})$ as θ does. By Lemma 2, $\tilde{\theta}$ stabilizes $t\tilde{\mathfrak{n}}$, so induces the desired element of $\operatorname{Aut}(\tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}})$.

We now prove that Θ is injective. Let θ in its kernel. By the definition of the group Γ, θ stabilizes $\mathfrak{h}$. Since the restrictions of the elements of $\Delta(I\mathfrak{b})$ span $\mathfrak{h}^*$, the restriction of θ to $\mathfrak{h}$ has to be the identity. In particular, θ acts trivially on $\Phi(I\mathfrak{b})$ and stabilizes each root space $(I\mathfrak{b})_\alpha$ for $\alpha \in \Phi(I\mathfrak{b})$. But θ stabilizes the set $\{X_\alpha : \alpha \in \Delta(I\mathfrak{b})\}$. Hence θ acts trivially on each $\tilde{\mathfrak{g}}_\alpha$ for $\alpha \in \Delta(I\mathfrak{b})$. Since $\tilde{\mathfrak{n}}$ is generated by the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \Delta(I\mathfrak{b})}$, the restriction of θ to $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is the identity map. Finally, θ is trivial and Θ is injective. □

Remark 15.

1. [1, Theorem 2] is the construction of an explicit order n automorphism of $\mathfrak{gl}^\epsilon_{n+}$. We can also interpret this automorphism in terms of the isomorphism $\mathfrak{gl}^\epsilon_{n+} \cong \tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}}$ of Theorem 4. Indeed, let θ be the cyclic automorphism of the extended Dynkin diagram in type A_ℓ and let $\tilde{\theta}$ be the automorphism of $\mathfrak{g}$ associated to θ as in Section 2.2. By Lemma 2 and the subsequent remark, $\tilde{\theta}$ induces an automorphism of $\tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}}$. Moreover, it is easily checked that the action on layer 1 in [1] is a cyclic permutation of the generators $(e_\alpha)_{\alpha \in \tilde{\Delta}}$.
2. Consider the trivial vector bundle $\underline{\mathcal{V}} := \mathcal{V} \times \mathbb{A}^1$ over $\mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[\epsilon])$. The Lie bracket $[\ ,\ ]_\epsilon$ endows $\underline{\mathcal{V}}$ with a structure of a Lie algebra bundle meaning that $[\ ,\ ]_\epsilon$ can be seen as a section of the vector bundle $\bigwedge^2 \underline{\mathcal{V}}^* \otimes \underline{\mathcal{V}}$ satisfying the Jaccobi identity. Consider the group $\operatorname{Aut}(\underline{\mathcal{V}}, [\ ,\ ]_\epsilon)$ consisting of automorphisms of the vector bundle $\underline{\mathcal{V}}$ respecting the Lie bracket pointwise. Let $\theta \in \operatorname{Aut}(\tilde{\mathcal{D}})$ and assume that the $\tilde{\theta} \in \operatorname{Aut}(\tilde{\mathfrak{g}})$ is $\mathbb{C}[t]$-linear (*i.e.* $\lambda = 1$ in Lemma 2). Then it is easy to check that $\tilde{\theta}$ induces an element of $\operatorname{Aut}(\underline{\mathcal{V}}, [\ ,\ ]_\epsilon)$. In other words, θ lifts to an $\mathbb{A}^1$-family of automorphisms over the $\mathbb{A}^1$-family of Lie algebras $\underline{\mathcal{V}}$.

4. Description of Aut($I\mathfrak{b}$)

In this section, we describe the structure of

$$\operatorname{Aut}(I\mathfrak{b}) = \{g \in \operatorname{GL}(I\mathfrak{b}) : \forall X, Y \in I\mathfrak{b} \qquad g([X,Y]) = [g(X), g(Y)]\}$$

[2] If $\mathfrak{g}$ is $\mathfrak{sl}_2$, Lemma 6 (2) does not apply. However, any permutation of $\tilde{\Delta}$ is an automorphism of the extended Dynkin diagram in this case.

in terms of the subgroups $U \cong \mathcal{M}_r(\mathbb{C})$, $\mathrm{Ad}(IB) \cong B \ltimes \mathfrak{g}/\mathfrak{b}$, $D \cong \mathbb{C}^*$ and $\Gamma \cong \mathrm{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ introduced in Section 3.

Observe that $\mathrm{Aut}(I\mathfrak{b})$ is a Zariski closed subgroup of the linear group $\mathrm{GL}(I\mathfrak{b})$.

Theorem 16. *We have the following decompositions*

$$\mathrm{Aut}(I\mathfrak{b}) = \Gamma \ltimes (D \ltimes (\mathrm{Ad}(IB) \times U)),$$

$$\mathrm{Aut}(\overline{I\mathfrak{b}}) = \Gamma \ltimes (D \ltimes (\mathrm{Ad}(IB)).$$

In particular, the neutral component is $\mathrm{Aut}(I\mathfrak{b})^\circ = D \ltimes (\mathrm{Ad}(IB) \times U)$ *and* Γ *can be seen as the component group of* $\mathrm{Aut}(I\mathfrak{b})$.

The result is a consequence of the lemmas provided below. Indeed, by Lemma 18, the four subgroups generate $\mathrm{Aut}(I\mathfrak{b})$. By Corollary 11(1) and Lemma 17 below, the subgroup generated by U and $\mathrm{Ad}(IB)$ is a direct product $U \times \mathrm{Ad}(IB)$. Then the structure of $\mathrm{Aut}(I\mathfrak{b})$ follows from Lemma 19. That of $\mathrm{Aut}(\overline{I\mathfrak{b}})$ follows the same lines, using Corollary 11(2). Note that we have identified Γ, $\mathrm{Ad}(IB)$ and D with their image under R, via Lemma 10.

Since D, $\mathrm{Ad}(IB)$ and U are connected and Γ is discrete, $\mathrm{Aut}(I\mathfrak{b}) = \bigsqcup_{g\in\Gamma} gD\mathrm{Ad}(IB)U$ is a finite disjoint union of irreducible subsets of the same dimension. They are thus the irreducible components of $\mathrm{Aut}(I\mathfrak{b})$ and the remaining statements of Theorem 16 follow.

Lemma 17. *The subgroups* U *and* $\mathrm{Ad}(IB)$ *are normal in* $\mathrm{Aut}(I\mathfrak{b})$*. Moreover,* $U \cap \mathrm{Ad}(IB) = \{\mathrm{Id}\}$.

Proof. Recall that $\mathrm{Ad}(IB)$ is generated by the exponentials of $\mathrm{ad}(x)$ with $x \in I\mathfrak{b}$. Then for any $\theta \in \mathrm{Aut}(I\mathfrak{b})$,

$$\theta\mathrm{Ad}(IB)\theta^{-1} = \theta\exp(I\mathfrak{b})\theta^{-1} = \exp(\theta(I\mathfrak{b})) = \exp(I\mathfrak{b}) = \mathrm{Ad}(IB).$$

Let $(b, f) \in IB$ and $h \in \mathfrak{h}$. Then $\mathrm{Ad}(b, f)(h) = b \cdot h + t\, b \cdot ([f, h] + \mathfrak{n})$. Assuming that $\mathrm{Ad}(b, f) = \bar{u} \in U$, we have $\mathrm{Ad}(b, f)(\mathfrak{h}) \subset \mathfrak{h} + \mathfrak{z}$ so $\mathrm{Ad}(b)(\mathfrak{h}) \subset \mathfrak{h}$, that is b belongs to the normalizer of $\mathfrak{h}$ in B, which turns to be T. In particular, $b \cdot [f, \mathfrak{h}] \subset \mathfrak{n}^-$ and $Ad(b, f)(\mathfrak{h}) \subset \mathfrak{h} + (\mathfrak{n} + t\mathfrak{n}^-)$. Hence $u = 0$ and finally $\mathrm{Ad}(IB) \cap U = \{\mathrm{Id}\}$. □

Lemma 18. *We have* $\mathrm{Aut}(I\mathfrak{b}) = \Gamma D\mathrm{Ad}(IB)U$ *and* $\mathrm{Aut}(\overline{I\mathfrak{b}}) = \Gamma D\mathrm{Ad}(IB)$.

Proof. Let $\theta \in \mathrm{Aut}(I\mathfrak{b})$. Since the two Cartan subalgebras $\mathfrak{c}$ and $\theta(\mathfrak{c})$ are Ad-conjugate (see [2, §3, n° 2, th. 1]), there exists $\theta_1 \in \mathrm{Ad}(IB)\theta$ which stabilizes $\mathfrak{c}$.

Then $\theta_1(\mathfrak{h})$ is complementary to the center $t\mathfrak{h} = \theta_1(t\mathfrak{h})$ in $\mathfrak{c}$. Thus, there exists $\theta_2 \in U\theta_1$ such that θ_2 stabilizes $\mathfrak{h}$.

Since θ_2 stabilizes $\mathfrak{c}$, it acts on $\Phi(I\mathfrak{b})$. Moreover, $I\mathfrak{b}^{(1)} = [I\mathfrak{b}, I\mathfrak{b}]$ and $I\mathfrak{b}^{(2)} = [I\mathfrak{b}^{(1)}, I\mathfrak{b}^{(1)}]$ are characteristic and stabilized by θ_2. So, Lemma 6 implies that θ_2 stabilizes $\Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})$ and hence $\Delta(I\mathfrak{b})$. Arguing as in the proof of Proposition 14, we show that the induced permutation is actually an automorphism of the extended Dynkin diagram. Thus there exists $\theta_3 \in \Gamma\theta_2$ with the additional property that the induced permutation on $\Delta(I\mathfrak{b})$ and thus on $\Phi(I\mathfrak{b})$ are trivial. Then θ_3 acts on each $(I\mathfrak{b})_\alpha$ for $\alpha \in \Delta(I\mathfrak{b})$.

Since Δ is a basis of $\mathfrak{h}^*$, one can find $h \in H \subset B \subset IB$ such that $\mathrm{Ad}(h) \circ \theta_3$ acts trivially on each $(I\mathfrak{b})_\alpha$ for $\alpha \in \Delta$. Moreover, D acts trivially on these roots spaces and with weight 1 on $(I\mathfrak{b})_{\alpha_0}$. This yields $\theta_4 \in D\mathrm{Ad}(H)\Gamma U \mathrm{Ad}(IB)\theta$ which acts trivially on $\mathfrak{h}$ and on each $(I\mathfrak{b})_\alpha$, $\alpha \in \Delta(I\mathfrak{b})$.

Recall now that $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is generated by the spaces $((I\mathfrak{b})_\alpha)_{\alpha\in\Delta(I\mathfrak{b})}$. Since θ_4 acts trivially on $\tilde{\mathfrak{n}}$ and on $\mathfrak{h}$, it has to be trivial. As a consequence, $\theta \in \mathrm{Ad}(IB)U\Gamma\mathrm{Ad}(H)D = \Gamma D\mathrm{Ad}(IB)U$, the last equality following from Lemma 17 and Corollary 11.

Recalling that $\Phi(I\mathfrak{b}) = \Phi(\overline{I\mathfrak{b}})$, the same proof applies for $\overline{I\mathfrak{b}}$ instead of $I\mathfrak{b}$, replacing $\mathfrak{c}$ by $\mathfrak{h}$ and skipping step from θ_1 to θ_2. □

Lemma 19. *The intersections $D \cap (\mathrm{Ad}(IB) \times U)$ and $\Gamma \cap (D \ltimes (\mathrm{Ad}(IB) \times U))$ are the trivial group $\{\mathrm{Id}\}$. Moreover, $(D \ltimes (\mathrm{Ad}(IB) \times U))$ is normal in $\mathrm{Aut}(I\mathfrak{b})$.*

Proof. Let $\tau \in \mathbb{C}^*$, $b \in B$, $f \in \mathfrak{g}/\mathfrak{n}$ and $u \in \mathrm{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$ such that the associated elements $\delta_\tau \in D$, $(b, f) \in IB$ and $\bar{u} \in U$ (see Section 3) satisfy $\delta_\tau = \mathrm{Ad}(b, f) \circ \bar{u}$. For $x \in \mathfrak{b}$, we have

$$x = \delta_\tau(x) = (\mathrm{Ad}(b, f) \circ \bar{u})(x) = \mathrm{Ad}(b, f)(x + u(x)) = b \cdot x + (b \cdot u(x) + tb \cdot ([f, x] + \mathfrak{n})).$$

In particular, $b \cdot x = x$ and, whenever $x \in \mathfrak{n}$, $b \cdot [f, x] = 0$ in $\mathfrak{g}/\mathfrak{n}$. So $b \in B$ centralizes $\mathfrak{b}$ and $\mathrm{ad}_\mathfrak{g} f$ normalizes $\mathfrak{n}$. As a consequence, $b = 1_B$, f is 0 in $\mathfrak{g}/\mathfrak{b}$ and $u = 0$. Thus the only element of $D \cap (\mathrm{Ad}(IB) \times U)$ is the trivial one.

Since $[I\mathfrak{b}, I\mathfrak{b}]$ is characteristic in $I\mathfrak{b}$, we have a natural group morphism $p : \mathrm{Aut}(I\mathfrak{b}) \to \mathrm{Aut}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}])$. From the description of $[I\mathfrak{b}, I\mathfrak{b}]$ in Lemma 5, it is straightforward that D, $\mathrm{Ad}(IB)$ and U are included in $\mathrm{Ker}(p)$ while $p_{|\Gamma}$ is injective. From Lemma 18, we then deduce that $D \ltimes (\mathrm{Ad}(IB) \times U) = \mathrm{Ker}(p)$ and the desired properties follow. □

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