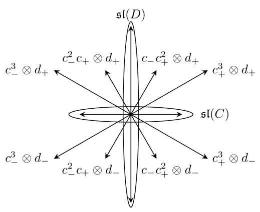
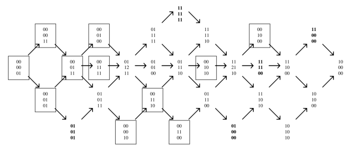
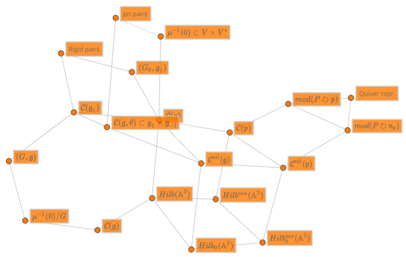
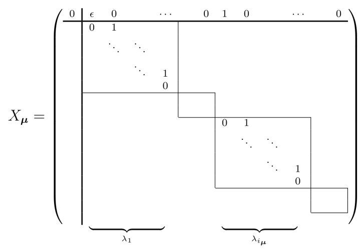




On commuting varieties and related topics

Autour des variétés commutantes





Mémoire d'HDR

par

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pour l'obtention de

l'*Habilitation à Diriger les Recherches*

Mathématiques

Soutenance prévue le Jeudi 3 décembre à 14H en ~~Salle A2 de la Faculté des Sciences et Techniques de l'UJM~~ Visioconférence

devant le jury composé de:

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[**Bu18**] M. Bulois, On the normality of the null-fiber of the moment map for θ- and tori representations, *Journal of Algebra*, **507** (2018), 502-524.

[**BE16**] M. Bulois and L. Evain, Nested punctual Hilbert schemes and commuting varieties of parabolic subalgebras, *Journal of Lie Theory*, **26** (2016), 497-533.

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2020, Nov 24th,

Michaël Bulois

[1]well, *almost never*

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1 Accessible Introduction

This dissertation can be labelled as lying in the junction of two general mathematical areas: algebra and geometry. More precisely, the studied objects are mainly *geometric objects* (commuting varieties, sheets, Hilbert schemes, ...) coming from *algebraic geometry* (such as the Hilbert schemes), from *symplectic geometry* (fibers of moment maps, symplectic reduction) or from *representation theory* (including algebraic groups, Lie algebras, quiver representations, invariant theory...). Most of the studied properties of these objects are properties stated in the language of algebraic geometry. This ranges from rather elementary descriptions, such as *characterisation of the irreducible components*, to more sophisticated ones, such as *smoothness*, *normality* and *reducedness of schemes*.

This dissertation is written in the perspective of getting the Habilitation degree. By nature, this rather self-centered exercise will be primarily focused on the author's works. If several selected other results are cited in order to set the context, no exhaustiveness was sought concerning the works related to the various mathematical questions addressed here.

1.1 Elementary concepts

Most of the objects of this dissertation are linked with *group actions on varieties*. These actions appears in many mathematical and extra-mathematical settings. We can cite group of symmetries (crystallographic groups, automorphism groups of varieties, Lorentz group and other invariance group of various mathematical models of theoretical physics...), elliptic curves (used in cryptography) as well as dynamical systems and symplectic geometry.

By *variety*, we mean an algebraic variety[2] and we will mostly speak of affine varieties, that is subsets of the affine space of dimension n defined as the vanishing locus of some polynomials (with n variables). For instance,

$$\{(x,y)|\, x^2+y^2=1\}, \qquad \text{the unit circle,} \tag{1.1}$$

$$\{(x,y)|\, y=0\}, \qquad \text{a line,} \tag{1.2}$$

$$\{(x,y)|\, y-x^2=0\}, \qquad \text{a parabola,} \tag{1.3}$$

$$\{(x,y)|\, xy=0\}, \qquad \text{the union of the two axis} \tag{1.4}$$

are subvarieties of the plane (the affine space of dimension 2). The example (1.4) is said to be *reducible* since it can be written as the union of two proper closed subvarieties $\{(x,y)|\, x=0\}$ and $\{(x,y)|\, y=0\}$ (the axes). On the other hand, examples (1.1), (1.2) and (1.3) are *irreducible*. Any algebraic variety

[2] *i.e.* a reduced separated scheme of finite type over a field. Varieties are not assumed to be irreducible in this dissertation

X has a unique decomposition as a union of *irreducible components* (maximal irreducible subvarieties of X). At each of its points, an algebraic variety can be either *smooth* or *singular*. Even if they have different definitions, two varieties X and Y can be *isomorphic*[3]. This is the case for the line and the parabola defined above[4]. Isomorphic varieties share the same geometric properties; *e.g.* there is a 1:1-correspondence between the irreducible components of both varieties. The geometric objects considered in this dissertation are defined over an algebraically closed field $\Bbbk$, which we will often assume to be of characteristic 0 (such as the field $\mathbb{C}$ of complex numbers).

An *action of a group on a variety* can be seen as a pair (G, V) where V is a variety and each element of the group G induces an automorphism of V (*i.e.* an invertible transformation of the variety V). The image of an element v of V by an element g of G will be denoted by $g \cdot v$. We say that G is an *algebraic group* if it is equipped with an algebraic variety structure, with multiplication and inverse map being both morphisms of varieties. The action we are interested in are *algebraic*, which means that the action map $\begin{cases} G \times V & \to & V \\ (g, v) & \mapsto & g \cdot v \end{cases}$ is a morphism of varieties. When V is a vector space and G acts linearly, we say that (G, V) is a *representation* . Among classical examples, we can cite

- an example in small dimension: the action of the torus $T_1 := \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ on $\mathbb{C}^2$ with weights $(1, -1)$, defined via $t \cdot (x, y) := (tx, t^{-1}y)$.
- the action on a vector space V of the group $GL(V)$ of invertible linear transformations of V.
- the action by conjugation of $GL_n = \{\text{invertible matrices of size } n\}$ on $\mathfrak{gl}_n = \{\text{square matrices of size } n\}$ via $g \cdot v := gvg^{-1}$ (change of basis formula).
- the action by conjugation of $SO_n = \{\text{orthogonal matrices of size } n\}$ on $S_n := \{\text{symmetric matrices of size } n\}$.

If (G, V) is an action, the *orbit* of an element $v \in V$ is the set $G \cdot v = \{g \cdot v |\, g \in G\}$. For instance, in the setting of the above described action $(T_1, \mathbb{C}^2)$, we get different types of orbits. When (x, y) satisfies $xy = c \neq 0$, we have $T_1 \cdot (x, y) = \{(a, b) |\, ab = c\}$ which is an orbit of dimension 1 (hyperbola); the element (x, y) is then said to be *semisimple* (or, more suitably, *polystable*) since its orbit is closed. When $x = 0$ and $y \neq 0$, $T_1 \cdot (0, y) = \{(0, b) |\, b \neq 0\}$ is a (non-closed) orbit of dimension 1; the element $(0, y)$ is said to be *nilpotent* (or, more suitably, *unstable*) since the orbit contains $(0, 0)$ in its closure. Similarly, when $x \neq 0$ and $y = 0$. The last orbit is $\{(0, 0)\}$ of dimension 0; the element being then both semisimple and nilpotent.

[3] *i.e.* there are *morphism of varieties* (*polynomial functions*), $X \to Y$ and $Y \to X$ whose compositions yield the identity morphisms on X and on Y

[4] via $(x, y) \to (x, y + x^2)$ and $(x, y) \to (x, y - x^2)$

The action $(GL_n, \mathfrak{gl}_n)$ is a particular case of the *adjoint action* on a Lie algebra, which is *reductive*[5] in this case. The Lie algebras should be seen as linearisations of algebraic groups. The theory of Lie algebras emerged in the second half of 19th century and has been constantly developed since. See [Ha00] for an historical account. The classification of the simple Lie algebras (which are the building blocks of the reductive Lie algebras and, in a weaker sense, of all Lie algebras) has been achieved by E. Cartan in the beginning of the 20th century. They can be sorted in 4 series (the so-called *classical types*) : A_n $(n \geqslant 1)$ which coincides with[6] $\mathfrak{gl}_{n+1}$ (modulo its center); B_n $(n \geqslant 2)$: $\mathfrak{so}_{2n+1}$; C_n $(n \geqslant 3)$: $\mathfrak{sp}_{2n}$; D_n $(n \geqslant 4)$: $\mathfrak{so}_{2n}$ and in five *exceptional* cases: E_6, E_7, E_8, F_4 et G_2.

Lie algebras are (unsurprisingly) algebraic objects. Namely, a Lie algebra is a vector space $\mathfrak{g}$ equipped with a *Lie bracket* (a "multiplication"), that is a skew-symmetric[7] bilinear map satisfying some specific associativity property (Jacobi identity). In type A, the bracket is defined on matrices via $[X, Y] = XY - YX$ and measures the defect of commutativity of the matrix multiplication. In reductive Lie algebras, the notions of semisimple and nilpotent elements have an algebraic definition which coincides with the above-given geometric definition. For instance, in type A, the semisimple matrices are the diagonalizable matrices, while nilpotent matrices are those for which some power vanish. In any reductive Lie algebra $\mathfrak{g}$, any element $x \in \mathfrak{g}$ has a unique *Jordan decomposition* $x = s + n$ where s is semisimple, n is nilpotent and $[s, n] = 0$. In particular, x is semisimple (respectively nilpotent) if and only if $n = 0$ (resp. $s = 0$).

1.2 Some more advanced geometry

Smooth points are the "nicest" points of a variety. At these points, varieties much behave like differential manifolds. For instance, if $x \in X$ and $y \in Y$ are smooth points and if X and Y have common dimension d locally around x and y, then X and Y are locally[8] isomorphic around x and y. This being said, a singularity can still have good propeties. For instance, it can be *normal* (ex: singularity at 0 of the cone $\{(x, y, z) | \, x^2 + y^2 - z^2 = 0\}$). Normality is a key

[5] reductive Lie algebras are direct sums of *abelian* Lie algebras (trivial bracket) and simple Lie algebras. At the other end of the classification of Lie algebras, we encounter *solvable* and *nilpotent* Lie algebras

[6] Classical Lie algebra are usually denoted with the same letters as the corresponding algebraic group, with lower case gothic letters. So $\mathfrak{so}_n$ (resp. $\mathfrak{sp}_n$) is the Lie algebra of the special orthogonal (resp. symplectic) group in dimension n.

[7] $[x, y] = -[y, x]$

[8] *étale-locally* here. *Étale topology* is a (Grothendieck) topology on algebro-geometric objects in which "local" has a meaning similar than in the complex manifold setting. On the other hand, we also often consider on algebro-geometric objects the *Zariski topology* which is a coarser topology. For instance, the Zariski-closed subsets of the n-dimensional space are the vanishing locus of sets of polynomials, so any non-empty open subset of $\Bbbk^n$ is dense.

property in algebraic geometry because of Zariski's main theorem which, for instance, allows to enhance set-theoretic properties to geometric properties:

Theorem 1.1. *Assume that Y is an irreducible variety. Then the following assertions are equivalent:*

- *Y is normal,*
- *Any morphism of irreducible varieties $f : X \to Y$ which is birational*[9] *with finite fibers, is in fact an isomorphism between X and a Zariski open subset of Y.*

A point $y \in Y$ is normal if and only if there exists a (Zariski) neighborhood of y in Y which is normal. In addition, at normal points, varieties are locally[10] irreducible[11], see *e.g.* [MO, §V.6].

At some points, we will encounter varieties defined by equations generating a possibly non-radical ideal. The correct object to consider in this situation is the corresponding *algebraic scheme*[12] defined by these equations. It is an algebro-geometric object whose points are in bijection with those of the variety (then called the associated *reduced* scheme) but which may carry additional infinitesimal information. If X is a scheme, the associated variety is denoted by X_{red}. By convention, when speaking about irreducible components of a scheme, we mean components of the associated variety[13].

We can illustrate this notion of scheme on single-point objects. In the plane $\Bbbk^2$, there is a unique variety whose only point is $(0,0)$. On the other hand, many subscheme of $\Bbbk^2$ are supported only at this single point. Let us mention those defined by the following equations

$$x = 0 = y, \quad \text{the reduced one} \tag{1.5}$$

$$x^2 = 0 = y - ax \quad (a \in \Bbbk) \quad \begin{array}{l}\text{the } \textit{length 2} \text{ subscheme at (0,0)} \\ \text{of the line with equation } y = ax\end{array} \tag{1.6}$$

The schemes of the form (1.6) "remember" the direction of the supporting line. A point of a scheme is said to be reduced if the scheme is reduced in a (Zariski) neighborhood of this point.

There are many desirable properties for varieties and schemes. They are reduced, smooth or normal when their locus of reduced (resp. smooth, resp.

[9] Alternatively, in our setting char $\Bbbk = 0$, we can replace "birational" by "bijection on dense (Zariski) open subsets of X and Y".

[10] *étale-locally* here.

[11] we then say that the variety is *unibranch* at these points

[12] *i.e.* separated scheme of finite type over $\Bbbk = \overline{\Bbbk}$

[13] *i.e.* we do not consider embeded components. Moreover, when we will look at reduced points of a given component Y of X, we will always be interested in reduced points of X belonging to the component Y.

normal) points coincides with the whole scheme. In general, these loci are known to form a (Zariski) open subset of the scheme. If $k \in \mathbb{N}$ is such that the singular locus of the scheme is of codimension at least $k+1$ in each irreducible component, we say that the scheme is *smooth in codimension* k or equivalently that it satisfies the condition (R_k). An irreducible component Y of a scheme X satisfies (R_0) in X if and only if it contains some smooth point of X if and only if it has at least one reduced point of X. The component Y is then said to be *generically reduced* or *generically smooth* in X.

We also say that a subscheme X of $\Bbbk^n$ is a *complete intersection* if and only if it can be defined by $(n - \dim X)$ equations in $\Bbbk^n$. It is then *equidimensional*, *i.e.* all the irreducible components of X share the same dimension. Using Serre's conditions (S_k) $(k \in \mathbb{N})$ which will not be defined here, we can write down the following graph of implications for the above mentioned possible properties on a given scheme:

$$\begin{array}{ccccc} \text{smooth} & \Longrightarrow & \text{normal} & \Longrightarrow & \text{reduced} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \forall k,\ (R_k) & & (S_2)+(R_1) & & (S_1)+(R_0) \end{array} \qquad \begin{array}{ccc} (S_{k+1}) \Rightarrow (S_k) & & (R_{k+1}) \Rightarrow (R_k) \\ & & \\ \begin{array}{c}\text{complete}\\ \text{intersection}\end{array} & \Longrightarrow & \forall k, (S_k) \end{array} \tag{1.7}$$

Another geometric feature of interest in this dissertation is *Invariant Theory*. Given a reductive group G acting on an affine variety V (*e.g.* if (G, V) is a representation), we can consider the *categorical quotient* $\pi : V \to V/\!\!/G$. It is a variety which has the following universal property: If $\varphi : V \to Y$ is a G-invariant morphism (*i.e.* $\forall g, v,\ \varphi(g \cdot v) = \varphi(v)$), then φ factors through π. Set-theoretically, we have

$$V/\!\!/G = \{\text{closed } G\text{-orbits in } V\}$$

and π sends $v \in V$ to *the* closed orbit lying in the closure of $G \cdot v$. For instance, if (G, V) is a representation and V^{nil} denotes the set of nilpotent elements in V, then $V^{nil} = \pi^{-1}(\pi(0))$. When all the orbits are closed, *e.g.* when G is finite, then the quotient $V/\!\!/G$ is also a set-theoretic one. We then say that it is a *geometric quotient* and we denote it by V/G. We refer to [PV91, §4] or [Kr84, II.3] for a more detailed exposition on these quotients.

For instance, in the above-described case of the action of T_1 on $\mathbb{C}^2$ with weights $(1, -1)$, we have $\mathbb{C}^2/\!\!/T_1 \cong \mathbb{C}$ with $\pi(x, y) = xy$. In particular, the two nilpotent orbits and the 0-orbit are all sent to 0 in the quotient. Following the general philosophy of *Geometric Invariant Theory* (GIT), we can consider open subsets of V to get better-behaved quotients. For instance, in the preceding example, we can set $U := \{(x, y) | x \neq 0\} \subset \mathbb{C}^2$. All the orbits in U become closed in U and we have a geometric quotient U/G. In this precise case, U/G

happen to be isomorphic to $\mathbb{C}^2/\!\!/G$. In general, such constructions might provide desingularisations of (open sets of) $V/\!\!/G$.

1.3 Some more advanced algebra

Reductive Lie algebras carry a lot of structure. A key notion involved in their classification is that of *Cartan subalgebra*. If $\mathfrak{g}$ is a reductive Lie algebra, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal subspace of semisimple elements. We have $[\mathfrak{h}, \mathfrak{h}] = 0$. In type A, an example is given by $\mathfrak{h} = \{\text{diagonal matrices}\}$. We define the *rank* of the Lie algebra $\mathfrak{g}$, by $\operatorname{rk} \mathfrak{g} := \dim \mathfrak{h}$. For instance $\operatorname{rk} \mathfrak{gl}_n = n$.

There exists several nested generalisations of the notion of reductive Lie algebra. These generalisations give rise to representations (G, V) where G is an algebraic group, V a vector space, and where the properties of the action imitate those of the adjoint action of an algebraic group on its Lie algebra. From more particular to more general settings, we can cite:

- *Symmetric Lie algebras.*
  A (reductive)[14] symmetric Lie algebra is a reductive Lie algebra $\mathfrak{g}$ equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$). It gives rise to an action of the form $(G_0, \mathfrak{g}_1)$ where G_0 is a connected algebraic group whose Lie algebra is $\mathfrak{g}_0$. The action (SO_n, S_n) mentioned in Section 1.1 is an example of such action[15].
  Simple symmetric Lie algebras have been classified by E. Cartan in 20 types, among which 8 series denoted by AI, AII, AIII, BI, DI, CI, CII, DIII and 12 exceptional cases EI, . . ., EIX, FI, FII and GI.

- *θ-groups.*
  Their definition is the same as the one of symmetric Lie algebras, except that the grading is a $\mathbb{Z}/m\mathbb{Z}$-grading (with $m \geqslant 1$, arbitrary). The studied action is still $(G_0, \mathfrak{g}_1)$. We refer to [Vi76] for main properties in this setting.

- *Polar representations.*
  A polar representation is a representation of a reductive group (G, V), such that V has big enough (*i.e.* of dimension $\dim V/\!\!/G$) subspaces $\mathfrak{c} \subset V$ of semisimple elements with locally parallel orbit (*i.e.* $\mathfrak{g} \cdot \mathfrak{c} = \mathfrak{g} \cdot c$ for some general element[16] $c \in \mathfrak{c}$). Such a subspace is called a Cartan subspace of the representation. Our main reference for polar actions is [DK85]

Any property proved at one level is valid in the previous settings (and in the setting of Lie algebras). In each of these settings, there is a notion of *Cartan*

[14]For simplicity of exposition, symmetric Lie algebras will always be assumed to be reductive
[15]here $\mathfrak{g} = \mathfrak{gl}_n$ is decomposed into antisymmetric and symmetric matrices
[16]by *general element* we will always mean *for all the elements lying in a suitable dense open subset*

subspace which is geometrically close to that of Cartan subalgebra. They are vector subspaces of V and any two of them are conjugate under the action of the group. As for ordinary Lie algebras, we define the *rank* of (G, V) via $\mathrm{rk}(G, V) = \dim \mathfrak{c}$ for any Cartan subspace $\mathfrak{c} \subset V$.

We now point out some desirable properties of a representation (G, V). We refer to [PV91, §8] for these properties and several related others. We say that the representation is *stable* if the general elements in V are semisimple (*i.e.* have closed orbits). This is equivalent to ask for the general fibers of the quotient map to be single orbits. Actions arising in the symmetric case (and thus adjoint action for reductive Lie algebras) are stable but this is not always the case for θ-groups (and thus neither for polar representations).

We say that the representation is *visible* if there are finitely many nilpotent orbits. Equivalently, each fiber of the quotient map is made of finitely many orbits. This is the case for any θ-groups but not necessarily for polar representations.

Lastly, the representation is said to be *locally free* if there are some orbits of maximal dimension $\dim G$. No adjoint action of a non-trivial Lie algebra satisfies this property but we can find some examples in each of the other settings.

We present a last class of representations, which are spaces of *quiver representations*. They are of the following form. Let Q be a (finite) oriented graph[17] with Q_0 its *vertex set* and Q_1 its *arrow set*. Let $(E_i)_{i \in Q_0}$ be a family of finite dimensional vector spaces. Let $G := \prod_{i \in Q_0} GL(E_i)$ and $V := \prod_{\varphi \in Q_1} \mathrm{Hom}(E_{t(\varphi)}, E_{h(\varphi)})$ where $t(\varphi)$ (resp. $h(\varphi)$) denotes the node on the "tail" (resp. "head") of $\varphi \in Q_1$. See [CB] for a detailed account on the subject.

By construction, this quiver setting has a lot to do with type A cases. For instance, the quotient $V /\!\!/ G$ can always be computed since invariants can be expressed by means of certain trace functions [LP90]. Also, the adjoint representation $(GL_n, \mathfrak{gl}_n)$ associated with the Lie algebras in type A can be realized as the above described representation associated with a quiver with one node and one loop, setting $\dim E = n$. Similarly, for symmetric Lie algebras of type AIII (resp. θ-group of inner type A), we consider the cyclic quiver with 2 nodes (resp. the cyclic quivers of an arbitrary length):

(1.8)

On the other hand, quivers yield a far more general class of representations. For instance, the quivers in (1.8) are all of *tame representation type* and it is manageable to describe the action of the group in a detailed manner[18]. However, most of the quivers (such as the double loop quiver) are of *wild*

[17] *i.e.* a *quiver*, since it contains *arrows*

[18] *e.g.* we can classify the orbits, as for any θ-group

representation type. They are linked to undecidable problems [Pr88] and a satisfactory classification of orbits is widely considered as intractable in these cases.

1.4 Objects of study

We now roughly present the content of this dissertation. A central object here is the **commuting variety**. Given a finite dimensional Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, its commuting variety is defined as

$$\mathcal{C}(\mathfrak{g})_{red} := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} |\, [x, y] = 0\} \tag{1.9}$$

The importance of this variety can be seen through the multiplicity of perspectives from which it can be considered.

First of all, it is an object defined in *algebraic* terms (*i.e.* with the algebraic operations of the Lie algebra). In type A, the commuting variety coincides with the set of pairs of matrices (X, Y) satisfying the commutativity relation $XY = YX$. In this dissertation, we will also consider some generalisations in reductive contexts (symmetric Lie algebras, θ-groups, polar representations) and in non-reductive ones (parabolic subalgebras of reductive Lie algebras). Additional constraints on the considered pairs can also be set (*e.g.* commuting nilpotent varieties, when we ask x and y to be nilpotent). As a first step, it is sometimes enlightening to describe such variety set-theoretically.

Second, it is also an object of *geometric* nature. It is an algebraic variety, namely an algebraic subset of the affine space $\mathfrak{g} \times \mathfrak{g}$. The relation $[x, y] = 0$ also defines a (possibly non-reduced) *scheme* $\mathcal{C}(\mathfrak{g})$. As a first rough description of these varieties, it is interesting to describe their irreducible components. Such description is one of the goals of Sections 2 and 3. Among finer properties that are also studied, we can cite the description of their *smooth*, *normal* or even *reduced* locus (cf. Conjecture 2.2).

In the study of the irreducible components of commuting varieties (Section 2), another important object of this dissertation appears: **sheets**. Given an algebraic action (G, V), the sheets are defined as the irreducible components of the sets of the form

$$V^{(m)} := \{x \in V |\, \dim G \cdot x = m\} \qquad (m \in \mathbb{N}).$$

We deal with the rich structure of these objects in the setting of symmetric Lie algebras in Section 4.

The commuting variety is also an example of a class of classical objects arising in the setting of *symplectic geometry*: zero fibers of *moment maps*. More precisely, given a representation (G, V), we define the symplectic double $V \times V^*$

and we can consider the moment map

$$\mu : \begin{cases} V \times V^* & \to & \mathfrak{g}^* \\ (v, \varphi) & \mapsto & \Big[g \mapsto \varphi(g \cdot v)\Big] \end{cases} . \tag{1.10}$$

When $V = \mathfrak{g}$ (adjoint action in the reductive Lie algebra case), we have $\mathcal{C}(\mathfrak{g}) \cong \mu^{-1}(0)$ via the Killing isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. In Section 2, we consider $\mathcal{C}(G, V)$ as $\mu^{-1}(0)$ for an arbitrary representation (G, V).

The commuting varieties are G-varieties so, in the setting of invariant theory mentioned in Section 1.2, it is natural to look at their quotients. Notably, in symplectic geometry, the categorical quotient $\mu^{-1}(0)/\!\!/G$ is a classical object called a **symplectic reduction**[19]. A classical type of desired result in such setting is the normality of the symplectic reduction. Such study is made in Section 5 for θ-groups and polar representations.

Lastly, we can also define *GIT quotients* of commuting varieties. In the $\mathfrak{gl}_n$-case, this allows to recover $\mathrm{Hilb}^n(\Bbbk^2)$, the punctual **Hilbert scheme** of the plane. Its points are the "length n" subschemes of the affine plane $\Bbbk^2$. For instance, for each $a \in \Bbbk$ the scheme described in (1.6) is a point of $\mathrm{Hilb}^2(\Bbbk^2)$; thus describing an affine line in $\mathrm{Hilb}^2(\Bbbk^2)$[20]. Hilbert schemes are natural objects studied in general algebraic geometry. In Section 6, we describe the connection with the commuting variety. We also focus on the so-called *nested Hilbert schemes* which parametrise pairs of subschemes $z_k \subset z_n$ of respective length k and n. It turns out to be related to the commuting variety of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{gl}_n$ (a non-reductive case). Considerations on the sheets of $\mathfrak{p}$ yield to the study of the representations of some specific quivers. There, the rich existing theory allows to derive important consequences for $\mathcal{C}(\mathfrak{p})$ and the corresponding Hilbert scheme.

In a last section, the annexe aims at providing **more context** for the existing directions of research related to the commuting variety. In a first part, we present how $\mathcal{C}(\mathfrak{g})$ fits in a poset of also studied subsets of $\mathfrak{g} \times \mathfrak{g}$. In a second part we present some other contemporary interesting directions of research which aim to study analogues of commuting varieties in different contexts.

[19]also known as *Hamiltonian reduction* or *Marsden-Weinstein reduction*

[20]and even in $\mathrm{Hilb}^2_0(\Bbbk^2)$, the Hilbert scheme paramterizing subschemes of $\Bbbk^2$ supported at the single point $(0, 0)$

2 Introduction to commuting varieties and sheets

Let $\mathfrak{g}$ be a finite dimensional Lie algebra defined over an algebraically closed field $\Bbbk$ of characteristic zero. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The commuting scheme $\mathcal{C}(\mathfrak{g})$ is defined as

$$\mathcal{C}(\mathfrak{g}) := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \,|\, [x, y] = 0\}. \tag{2.1}$$

We also consider the commuting variety $\mathcal{C}(\mathfrak{g})_{red}$, the reduced scheme associated to $\mathcal{C}(\mathfrak{g})$.

When $\mathfrak{g}$ is reductive, $\mathfrak{h}$ is abelian, so $\mathfrak{h} \times \mathfrak{h} \subset \mathcal{C}(\mathfrak{g})$. The following theorem is a seminal result of Richardson[21]

Theorem 2.1. *[Ri79]. Assume $\mathfrak{g}$ is reductive, then $\mathcal{C}(\mathfrak{g})_{red} = \overline{G \cdot (\mathfrak{h} \times \mathfrak{h})}$. In particular, $\mathcal{C}(\mathfrak{g})$ is irreducible of dimension* $\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g}$.

The following long standing conjecture (see e.g. [Va94, §9.3]) aims at describing finer geometry of the commuting variety

Conjecture 2.2. *The commuting scheme $\mathcal{C}(\mathfrak{g})$ is reduced and normal.*

Several generalizations of the commuting scheme are possible. Some of them are of the form $\mathcal{C}(G, V) := \mu^{-1}(0)$ for some representation (G, V), in the notation of the end of Section 1.4. That is

$$\mathcal{C}(G, V) = \{(x, \varphi) \in V \times V^* \,|\, \varphi \in [\mathfrak{g}, x]^{\perp}\}. \tag{2.2}$$

When $\mathfrak{g}$ is reductive, we have $\mathcal{C}(\mathfrak{g}) \cong \mathcal{C}(G, \mathfrak{g})$ where $(G, \mathfrak{g})$ is the adjoint action, using the Killing isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. In the special case $(G, V) = (SO_n, S_n)$, this corresponds to the set of pairs of commuting symmetric matrices. Another possible generalization is

$$\mathcal{C}'(G, V) := \{(g, v) \in \mathfrak{g} \times V \,|\, g \cdot v = 0\}. \tag{2.3}$$

In a different philosophy from Theorem 2.1, we are also interested in the nilpotent commuting varieties which focus on the least semisimple commuting pairs. For instance

$$\mathcal{C}^{nil}(\mathfrak{g}) := \{(x, y) \in \mathcal{C}(\mathfrak{g}) \,|\, x, y \text{ nilpotent}\}^{22} \tag{2.4}$$

[21] previously proved by Gerstenhaber in type A [Ge61]

[22] there are several natural scheme structures encoding the nilpotency condition here. For instance, one can consider the scheme-theoretic intersection of $\mathcal{C}(\mathfrak{g})$ with $\mathfrak{g}^{nil} \times \mathfrak{g}^{nil}$ where $\mathfrak{g}^{nil}$ is the (reduced) nilpotent cone. In $\mathfrak{gl}_n$, we can also consider sets of defining equations like $x^n = y^n = 0$ or even $x^n = x^{n-1}y = \cdots = xy^{n-1} = y^n = 0$. Most of the time, we look at properties of the underlying variety such as irreducible components or dimension, so the scheme structure play no role. The only exception is in Theorem 6.3 (2), where the last mentioned scheme structure is used.

<table>
<tr><th rowspan="2">setting</th><th rowspan="2">symbol</th><th colspan="4">equivalent definition</th></tr>
<tr><th>dimension</th><th>irreducibility?</th><th>references</th><th>equidimensionality?</th></tr>
<tr><td rowspan="4">$\mathfrak{g}$ reductive Lie algebra</td><td rowspan="2">$\mathcal{C}(\mathfrak{g})$</td><td colspan="4">$\mathcal{C}(G,\mathfrak{g}) = \mathcal{C}'(G,\mathfrak{g}) = \{(x,y) \in \mathfrak{g}\times\mathfrak{g} \mid [x,y]=0\}$</td></tr>
<tr><td>$\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g}$</td><td>yes</td><td>[Ri79]</td><td>/</td></tr>
<tr><td rowspan="2">$\mathcal{C}^{nil}(\mathfrak{g})$</td><td colspan="4">$\{(x,y) \in \mathcal{C}(\mathfrak{g}) \mid x,y \text{ nilpotents}\} = \mathcal{C}(\mathfrak{g}) \cap \mathfrak{g}^{nil}\times\mathfrak{g}^{nil}$</td></tr>
<tr><td>$\dim\mathfrak{g}$</td><td>depends (C)</td><td>[Pr03]</td><td>yes</td></tr>
<tr><td rowspan="4">(G,V) representation</td><td rowspan="2">$\mathcal{C}(G,V)$</td><td colspan="4">$\{(x,\varphi) \in V\times V^* \mid \varphi \in [\mathfrak{g},x]^\perp\}$</td></tr>
<tr><td>$\dim V + \mathrm{mod}(G,V)$</td><td>depends</td><td>[Pa94, BLLT17, HSS20]</td><td>depends</td></tr>
<tr><td rowspan="2">$\mathcal{C}'(G,V)$</td><td colspan="4">$\{(g,x) \in \mathfrak{g}\times V \mid g\cdot x = 0\}$</td></tr>
<tr><td>$\dim G + \mathrm{mod}(G,V)$</td><td>depends</td><td>[Pa94]</td><td>depends</td></tr>
<tr><td rowspan="2">$\mathfrak{g} = \bigoplus_{\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$
reductive θ-group</td><td rowspan="2">$\mathcal{C}(G_0,\mathfrak{g}_1)$</td><td colspan="4">$\{(x,y) \in \mathfrak{g}_1\times\mathfrak{g}_{-1} \mid [x,y]=0\} = \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}_1\times\mathfrak{g}_{-1})$</td></tr>
<tr><td>$\dim\mathfrak{g}_1 + \mathrm{rk}(G_0,\mathfrak{g}_1)$</td><td>depends</td><td>[Pa94, Bu18]</td><td>$\Leftrightarrow$ irred</td></tr>
<tr><td rowspan="4">$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$
reductive symmetric
Lie algebra</td><td rowspan="2">$\mathcal{C}(G_0,\mathfrak{g}_1)$</td><td colspan="4">$\{(x,y) \in \mathfrak{g}_1\times\mathfrak{g}_1 \mid [x,y]=0\} = \mathcal{C}(\mathfrak{g}) \cap (\mathfrak{g}_1\times\mathfrak{g}_1)$</td></tr>
<tr><td>$\dim\mathfrak{g}_1 + \mathrm{rk}(G_0,\mathfrak{g}_1)$</td><td>depends (C)</td><td>[SY06, PY07, Bu11b]</td><td>$\Leftrightarrow$ irred</td></tr>
<tr><td rowspan="2">$\mathcal{C}^{\mathrm{nil}}(G_0,\mathfrak{g}_1)$</td><td colspan="4">$\mathcal{C}^{nil}(\mathfrak{g}) \cap \mathfrak{g}_1\times\mathfrak{g}_1 = \mathcal{C}(G_0,\mathfrak{g}_1) \cap \mathfrak{g}^{nil}\times\mathfrak{g}^{nil}$</td></tr>
<tr><td>$\dim\mathfrak{g}_1$</td><td>depends (PC)</td><td>[Bu09]</td><td>conjecturally yes (PC)</td></tr>
<tr><td rowspan="4">$\mathfrak{p}$ parabolic
Lie subalgebra
of a reductive one</td><td rowspan="2">$\mathcal{C}(\mathfrak{p})$</td><td colspan="4">$= \mathcal{C}'(P,\mathfrak{p}) = \{(x,y) \in \mathfrak{p}\times\mathfrak{p} \mid [x,y]=0\}$</td></tr>
<tr><td>$\dim\mathfrak{p} + \mathrm{mod}(P,\mathfrak{p})$</td><td>depends ((C) for Borel)</td><td>[GG18, BB19]</td><td>depends</td></tr>
<tr><td rowspan="2">$\mathcal{C}^{nil}(\mathfrak{p})$</td><td colspan="4">$= (\mathcal{C}')^{nil}(P,\mathfrak{p}) = \mathcal{C}(\mathfrak{p}) \cap \mathfrak{p}^{nil}\times\mathfrak{p}^{nil}$</td></tr>
<tr><td>$\dim\mathfrak{p} + \mathrm{mod}(P,\mathfrak{p}^{nil})$</td><td>depends</td><td>[BB19]</td><td>depends ((PC) for $\mathfrak{gl}_n$)</td></tr>
</table>

Table 1: Various commuting varieties

In the column "irreducibility", the mention "depends" means that there are both examples of irreducible and non-irreducible commuting varieties. Same for equidimensionality.

The mention (C) means that there exists a classification of the cases where the variety is irreducible (resp. equidimensional) or not.

The mention (PC) means that a partial such classification exists, say with at least half of the cases treated, whatever this means.

A summary of the commuting varieties/schemes mentioned in this thesis is given in Table 1

The study of these commuting varieties is eased by considerations on sheets [Pa94, SY06]. In the setting of an algebraic action (G, V) of an algebraic group G on a variety V, the latter can be stratified by orbit dimension. Namely, given $m \in \mathbb{N}$, we consider the following locally closed subset of V:

$$V^{(m)} := \{x \in V | \dim G \cdot x = m\} \tag{2.5}$$

The **sheets** are the irreducible components of such $V^{(m)}$. The modality of a sheet S is defined as $\operatorname{mod} S := \dim S - m$. Intuitively, this corresponds to the number of continuous parameters needed to classify the orbits in S. This is also the dimension of the quotient S/G, whenever it exists. We also define the modality of (G, V) as $\operatorname{mod}(G, V) := \max_S(\operatorname{mod} S)$ where the maximum is taken over the set of sheets.

Theorem 2.3. *Let (G, V) be a representation*

(i) If S is a sheet of V, $\mathcal{C}_S(G, V) := \{(x, y) \in \mathcal{C}(G, V) | x \in S\}$ is an irreducible subvariety of $\mathcal{C}(G, V)$ of dimension $\dim V + \operatorname{mod} S$.

(ii) Each irreducible component of $\mathcal{C}(G, V)$ is of the form $\overline{\mathcal{C}_S(G, V)}$ for some sheet S of V.
In particular, $\dim \mathcal{C}(G, V) = \dim V + \operatorname{mod}(G, V)$

Proof. The conclusion in (ii) was already present in [Pa94]. The irreducibility statement in (i) follows mainly from [SY06, Lemma 2.2]. Let us give a stronger version of this last result.

Lemma 2.4. *Let X, Y be varieties and let E be a (locally closed) subvariety of $X \times Y$. Assume moreover that*

1. *$\mathrm{pr}_1(E) \subset X$ is irreducible*[23].
2. *The non-empty fibers of $(\mathrm{pr}_1)_{|E}$ are all irreducible of fixed dimension r.*
3. *There exists $y_0 \in Y$ such that $\mathrm{pr}_1(E) \times \{y_0\} \subset E$.*

Then E is irreducible

Proof. Let F be an irreducible component of E of maximal dimension. Then for a general $x \in \mathrm{pr}_1(F)$, $\dim E = \dim F = \dim \mathrm{pr}_1(F) + \dim(\mathrm{pr}_1)^{-1}_{|F}(x) \leqslant \dim \mathrm{pr}_1(E) + r = \dim E$. So $\dim \mathrm{pr}_1(F) = \dim \mathrm{pr}_1(E)$ and the non-empty fibers of $(\mathrm{pr}_1)_{|F}$ are of dimension r. Since F is closed in E, for any $x \in \mathrm{pr}_1(F)$, assumption 2 implies that $(\mathrm{pr}_1)^{-1}_{|F}(x) = (\mathrm{pr}_1)^{-1}_{|E}(x)$. In particular, $\mathrm{pr}_1(F) \times \{y_0\} \subset F$ so, by assumptions 3 and 1, $\mathrm{pr}_1(F) = \mathrm{pr}_1(E)$. Thus $F = E$. □

[23] $\mathrm{pr}_1(E)$ might not be a subvariety of X. Neverthless, many notions such as irreducibility and dimension still make sense for such subsets, see [TY05, §1].

End of proof of the theorem. The sheets are irreducible locally closed subvarieties of V. Thus each $\mathcal{C}_S(G,V) = (\mathcal{C}(G,V) \cap (S \times V^*))_{red}$ is locally closed. We will apply Lemma 2.4 to $E = \mathcal{C}_S(G,V)$, $X = V$, $Y = V^*$ and $y_0 = 0$. In this setting each $(\mathrm{pr}_1)^{-1}_{|E}(x)$ $(x \in S)$ is the orthogonal in V^* of the vector space $[\mathfrak{g}, x] \subset V$. It is thus a vector space of fixed constant dimension $\dim V - \dim[\mathfrak{g}, x] = \dim V - m_S$. This proves (i). Statement (ii) follows since $\mathcal{C}(G,V)_{red}$ coincides with the finite union $\bigcup_S \mathcal{C}_S(G,V)$. □

An easy adaptation of the proof of the previous theorem yields the following

Remark 2.5. (i) When the notion makes sense V^{nil}, the set of nilpotent elements in V, is G-stable. If S is a sheet of V^{nil}, then the subvariety $\mathcal{C}_S^{nil}(G,V) := \{(x,y) \in \mathcal{C}^{nil}(G,V) | \, x \in S\}$ is of dimension at most $\dim V + \mathrm{mod}\, S$ with equality if and only if S is a sheet in V^{nil} of *distinguished* elements[24]. In this last case, $\mathcal{C}_S^{nil}(G,V)$ is also irreducible.

In particular, the sheets made of distinguished elements with maximal modality, when they exist, are in bijection with the irreducible components of maximal dimension $\dim V + \mathrm{mod}(G, V^{nil})$ in $\mathcal{C}^{nil}(G,V)$. Note that in many classical settings, including the representation associated to any θ-group, V^{nil} is a finite union of orbits so $\mathrm{mod}(G, V^{nil}) = 0$.

(ii) If we replace $\mathcal{C}(G,V)$ by $\mathcal{C}'(G,V)$, we can define $\mathcal{C}'_S(G,V) := \{(x,y) \in \mathcal{C}'(G,V) | y \in S\}$. Then the same statements as in Theorem 2.3 (i) and (ii) hold for $\mathcal{C}'(G,V)$ with each occurrence of $\dim V$ replaced by $\dim G$.

Same for $(\mathcal{C}')^{nil}(G,V)$ when this makes sense, e.g. for $\mathcal{C}^{nil}(\mathfrak{p})$ when $\mathfrak{p}$ is a parabolic subalgebra of a semisimple Lie algebra.

For the commuting varieties under consideration, we often consider the *principal component* of the variety, which corresponds to $\overline{\mathcal{C}_{S_{reg}}(G,V)}$ where S_{reg} is the sheet of regular elements (*i.e.* whose orbits are of maximal dimension). This is the irreducible component of $\mathcal{C}(G,V)$ dominating V via pr_1. Similar principal components can be defined in the other settings of Table 1[25].

Proving the irreducibility of a commuting variety often amounts to show that there is no other component. For instance, a crucial step in the proof of Theorem 2.1 is that, in a given sheet $S \subset \mathfrak{g}$, the general elements are either semisimple or commute with non-trivial semisimple elements.

Here is a list of results obtained by various authors and directly related to my works presented in this dissertation.

Theorem 2.6. *1. If $\mathfrak{g}$ is a reductive Lie algebra then $\mathcal{C}(\mathfrak{g})$ satisfies (R_2). [Po08]*

[24] $x \in V^{nil}$ is said to be distinguished if and only if $(x,y) \in \mathcal{C}(G,V) \Rightarrow y \in (V^*)^{nil}$

[25] with possibly several such components in nilpotent commuting varieties when V^{nil} is not irreducible

2. *If $\mathfrak{g}$ is a reductive Lie algebra then the irreducible components of $\mathcal{C}^{nil}(\mathfrak{g})$ are precisely those corresponding to distinguished nilpotent orbits. In particular $\mathcal{C}^{nil}(\mathfrak{g})$ is equidimensional of dimension* $\dim \mathfrak{g}$*. [Pr03]*

3. *If $(G_0, \mathfrak{g}_1)$ is an action coming from a symmetric Lie algebra, then the principal component of $\mathcal{C}(G_0, \mathfrak{g}_1)$ is its unique component of maximal dimension* $\dim \mathfrak{g}_1 + \mathrm{rk}(\mathfrak{g}_0, \mathfrak{g}_1)$ *[SY06]. A classification of the cases when $\mathcal{C}(G_0, \mathfrak{g}_1)$ is irreducible or not is given in* 17 *cases (among* 20*) in [PY07] and references therein.*

4. *If $(G_0, \mathfrak{g}_1)$ is an action coming from a symmetric Lie algebra, in* 14 *cases among* 20*, the irreducible components of $\mathcal{C}^{nil}(\mathfrak{g}_1)$ are precisely those corresponding to ($\mathfrak{g}_1$-)distinguished nilpotent orbits. In particular $\mathcal{C}^{nil}(G_0, \mathfrak{g}_1)$ is of pure dimension* $\dim \mathfrak{g}_1$*. Conjecturally, this still holds true in the* 6 *remaining cases [Bu09] (Phd results)*

5. *If (G, V) is such that G is reductive and the representation is visible, stable and locally free, then $\mathcal{C}(G, V)$ is an irreducible and reduced complete intersection. [Pa94]*
   In particular, it is normal if and only if it is smooth in codimension 1 *(see* (1.7)*).*

6. *If $\mathfrak{p}$ is a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$, then the minimal possible dimension of the irreducible components of $\mathcal{C}(\mathfrak{p})$ is* $\dim \mathfrak{p} + \mathrm{rk}\,\mathfrak{g}$*, which is the dimension of the principal component.*
   The cases when $\mathcal{C}(\mathfrak{p})$ is irreducible are characterised via a numerical criterion involving modality of some varieties of nilpotent elements[26]*.*
   When $\mathfrak{p}$ is a Borel subalgebra of $\mathfrak{g}$, the classification into irreducible and reducible cases is completed. [GG18]

[26] including $\mathrm{mod}(P, \mathfrak{p}^{nil})$, but also similar modalities in Levi factors of $\mathfrak{g}$

3 Smooth locus and irreducible components in the symmetric case

Following Theorem 2.6 (1), the author studied in [Bu11b] the smooth locus of $\mathcal{C}(G_0, \mathfrak{g}_1)$ where $(G_0, \mathfrak{g}_1)$ is the adjoint action associated with a reductive symmetric Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

More precisely, V. Popov showed in [Po08] that the singular locus of the commuting scheme of a reductive Lie algebra coincides with the irregular locus, that is the set of elements whose orbit is not of maximal dimension $\dim G - \dim \mathfrak{h}$:

$$\mathcal{C}^{irr}(\mathfrak{g}) := \{(x, y) \in \mathcal{C}(\mathfrak{g}) | \dim \mathfrak{g}^{x,y} > \dim \mathfrak{h}\} \tag{3.1}$$

where $\mathfrak{g}^{x,y}$ is the stabiliser of (x, y) in $\mathfrak{g}$. Then, looking at some key decomposition classes (cf. beginning of Section 4) and arguing by induction on the rank of $\mathfrak{g}$, he classified these pairs in each case. As a result, he showed that the codimension of $\mathcal{C}^{irr}(\mathfrak{g})$ in $\mathcal{C}(\mathfrak{g})$ is 2, 3 or 4, depending of the type of the Lie algebra. In particular, $\mathcal{C}(\mathfrak{g})_{red}$ is at least smooth respectively in codimension 1, 2 and 3.[27]

Turning to the symmetric case, it follows from Theorem 2.6 (3) that the commuting variety is not irreducible in several cases. The natural generalisation of Popov's result in this setting requires to focus on the principal component of $\mathcal{C}(G_0, \mathfrak{g}_1)$. This yielded:

Theorem 3.1. *[Bu11b] The principal component of $\mathcal{C}(G_0, \mathfrak{g}_1)$ satisfies (R_2) in $\mathcal{C}(G_0, \mathfrak{g}_1)$* [28].

This was already proved in the locally free case[29] by D. Panyushev [Pa94] in connection with Theorem 2.6 (5).

A difficulty arising in the present general symmetric setting is the initialization step of the induction argument since, for instance, there are 7 types of symmetric Lie algebras of rank 1, including several series. One consequence is that there no longer exists any uniform upper bound for the codimension of the irregular locus. Moreover, this precise codimension could only be computed in 23 (sub-)cases among 30.

Nevertheless, part of the argumentation still makes sense for all the components of $\mathcal{C}(G_0, \mathfrak{g}_1)$. This can be stated as follows. Assume that $\overline{\mathcal{C}_S(G_0, \mathfrak{g}_1)}$ is an irreducible component of $\mathcal{C}(G_0, \mathfrak{g}_1)$ for some sheet S (cf. Theorem 2.3). Then, in this component, the smooth points of the whole commuting scheme are the pairs (x, y) satisfying $\dim \mathfrak{g}_0^{x,y} = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 + \operatorname{mod} S$[30]. By combining the

[27] and no more if $\mathcal{C}(\mathfrak{g})$ happens to be reduced

[28] *i.e.* the singular points of $\mathcal{C}(G_0, \mathfrak{g}_1)$ belonging to the principal component form a closed subset of codimension at least 2 in this component.

[29] Also known as the *maximal rank* case, since this is equivalent to the condition $\operatorname{rk}(\mathfrak{g}_0, \mathfrak{g}_1) = \operatorname{rk}(\mathfrak{g})$

[30] which is $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 + \operatorname{rk}(\mathfrak{g}_0, \mathfrak{g}_1)$ for the principal component

increasingness of $\dim \overline{\mathcal{C}_S(G_0,\mathfrak{g}_1)}$ in mod S with the upper semi-continuity of the map $(x,y) \mapsto \mathfrak{g}_0^{x,y}$, we can thus detect the components admitting smooth points simply by looking at $\dim \mathfrak{g}_0^{x,y}$.

After reformulation, this can be stated in terms of *rigid pairs*:

Definition 3.2. Let $(x,y) \in \mathcal{C}(G_0,\mathfrak{g}_1)$. We say that (x,y) is a rigid pair if $\dim \mathfrak{g}_0^{x,y} = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1$.

Theorem 3.3. *[Bu11b] In the symmetric case,*

1. *There is a bijection between irreducible components which are generically reduced in $\mathcal{C}(G_0,\mathfrak{g}_1)$ [31] and conjugacy classes of pairs $(\mathfrak{l},(x,y))$ where*
   - *$\mathfrak{l}$ is the centralizer in $\mathfrak{g}$ of a semisimple element of $\mathfrak{g}_1$*
   - *(x,y) is a rigid pair in the odd part of the derived algebra of $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$*

   The component corresponding to $(\mathfrak{l},(x,y))$ is then $\overline{G_0 \cdot (\mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l}) + x, \mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l}) + y)}$ where $\mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l})$ is the centralizer of $\mathfrak{l}$ in $\mathfrak{g}_1$[32].
2. *The classification of cases when $\mathcal{C}(G_0,\mathfrak{g}_1)$ is irreducible has been achieved (cf. Theorem 2.6 (3)).*
   Moreover, all the reducible cases are shown to admit such a generically reduced component different from the principal component.

Further elements of proof. In order to get (1), one needs to refine the decomposition $\mathcal{C}(G_0,\mathfrak{g}_1)_{red} = \bigcup \mathcal{C}_S(G_0,\mathfrak{g}_1)$. Namely, we replace the sheets by decomposition classes, see Section 4. Each class is of the form $J := G_0 \cdot (\mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l})^\bullet + x)$ for some centralizer $\mathfrak{l}$ of a semisimple element of $\mathfrak{g}_1$ and some nilpotent element $x \in \mathfrak{l}_1$.

The set of elements commuting with any element of $\mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l})^\bullet + x$ is $\mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l}) + \mathfrak{l}_1^x$. Also, $\mathfrak{g}_0^{(x,y)} = \mathfrak{c}_{\mathfrak{g}_0}(\mathfrak{l}) + \mathfrak{l}_0^{(x,y)}$ Thus studying commuting pairs (z_1,z_2) with $z_1 \in J$ amounts mostly to study commuting pairs in $\mathfrak{l}_1$ of the form (x,y). This explains where the subalgebras $\mathfrak{l}$ in (1) come from and partly why the argument sketched above Definition 3.2 translates as stated.

The given form of the component associated to $(\mathfrak{l},(x,y))$ follows from the following remarkable fact concerning rigid pairs: If (x,y) is rigid in $\mathfrak{l}$ then $L_0.y$ is dense in $\mathfrak{l}_1^x$.

Concerning the first part of (2), the author then simply found in each of the 3 remaining cases some non-trivial rigid pairs, thus showing the existence of irreducible components other than the principal one. □

Comments and Perspectives

From Theorem 3.3, it is natural to look for the following weaker analogue of Conjecture 2.2 in the symmetric case.

[31] *i.e.* components admitting smooth points of the commuting scheme, as defined in the introduction.

[32] *i.e.* $\mathfrak{c}_{\mathfrak{g}_1}(\mathfrak{l})$ is the odd part of the center of the Levi $\mathfrak{l}$

Conjecture 3.4. *[Bu11b] When* $(G_0, \mathfrak{g}_1)$ *is the representation arising from a symmetric Lie algebra, then all the components of* $\mathcal{C}(G_0, \mathfrak{g}_1)$ *are generically reduced in* $\mathcal{C}(G_0, \mathfrak{g}_1)$[33]*.*

It is worth noting from the subsequent Example 5.7 that $\mathcal{C}(G_0, \mathfrak{g}_1)$ is not reduced in general. It is thus hard to ask for a stronger statement than Conjecture 3.4 concerning reducedness.

In practice, we can list the rigid pairs for a given symmetric Lie algebra using probabilistic tests. More precisely, for each nilpotent orbit $G_0 \cdot x \subset \mathfrak{g}_1$ the set of $y \in \mathfrak{g}_1^x$ such that (x, y) is a rigid pair is open in $\mathfrak{g}_1^x$. Thus choosing y randomly, we should fall in this open set, whenever it is non-empty, with a high rate of success. Such algorithm provides a list of conjugacy classes of rigid pairs which is likely to be complete. The uncertainty can be shrunk by testing multiple choices for y.

In order to show that the list provided by the above mentioned algorithm is complete, it is sufficient to show that the orbits $G_0 \cdot x$, expected not to be involved in any rigid pair (x, y), do not give rise to an irreducible component of $\mathcal{C}(G_0, \mathfrak{g}_1)$, see Theorem 3.3.

Using these methods the author computed[34] the complete list of rigid pairs in several low-rank cases, thus checking Conjecture 3.4 in these cases. Notably, this has been done for all the exceptional cases. Also, the previously known components exhibited in [PY07] all arise from rigid pairs as in Theorem 3.3.

For the classical cases, there are only three series [35] where $\mathcal{C}(G_0, \mathfrak{g}_1)$ is not always irreducible. In these cases several **open questions remain** such as:

- The classification of rigid pairs.
- The validity of Conjecture 3.4.

In order to answer to the first question, one could take inspiration from the various classifications of pairs mentioned in the Annexe 7[36]. Preliminary computations suggest that this classification would be quite involved. For instance, no "grid-like" description, like the skew-graphs of [EP01], apply to the description of some rigid pairs, even in type AIII.

Concerning the conjecture, we could then try to show case by case that, given a sheet S for which $\mathcal{C}_S(G_0, \mathfrak{g}_1)$ has no smooth points of $\mathcal{C}(G_0, \mathfrak{g}_1)$[37], then $\mathcal{C}_S(G_0, \mathfrak{g}_1)$ is not an irreducible component of $\mathcal{C}(G_0, \mathfrak{g}_1)$. Indeed, there are many requirements on a sheet S for $\mathcal{C}_S(G_0, \mathfrak{g}_1)$ to be an irreducible component. For instance, many cases of irreducibility were shown by ruling out this possibility

[33] *i.e.* $\mathcal{C}(G_0, \mathfrak{g}_1)$ is generically reduced
[34] unpublished work
[35] AIII, CII and DIII
[36] All the more so as rigid pairs turn out to be nilpotent pairs
[37] *i.e.* $\mathcal{C}(G_0, \mathfrak{g}_1)$ does not correspond to a pair $(\mathfrak{l}, (x, y))$ with (x, y) rigid, in Theorem 3.3

for all the sheets apart from the regular one. However, there is no evidence that the available discarding arguments would be enough to conclude here.

4 Sheets

We now look at sheets in reductive Lie algebras and reductive symmetric Lie algebras. That is, we consider an action (M, V) of the form $(G, \mathfrak{g})$ or $(G_0, \mathfrak{g}_1)$ depending on the context. It is then useful to consider the so-called *decomposition classes*. Each of these classes gathers the elements with similar Jordan decomposition. Given an element $x \in V$, we denote its Jordan decomposition by $x = x_s + x_n$ where x_s is its semisimple part and x_n its nilpotent part. We refer to [TY05, §39] for most of the known properties on decomposition classes[38].

Definition 4.1. The decomposition class of an element $x \in V$ is

$$\begin{aligned} J(x) &:= \{y \in V \mid \mathfrak{g}^y \text{ is conjugate to } \mathfrak{g}^x\} \\ &= M.\{y_s + y_n \mid \mathfrak{g}^{y_s} = \mathfrak{g}^{x_s} \text{ and } M^{x_s} \cdot x_n = M^{x_s} \cdot y_n\} \end{aligned}$$

In particular, decomposition classes are classified by conjugacy classes of pairs of the form $(\mathfrak{l}, \mathcal{O})$ where

- $\mathfrak{l}$ is a standard Levi factor *arising from* V, that is a centralizer in $\mathfrak{g}$ of a semisimple element of V.[39]
- $\mathcal{O}$ is an $(M \cap L)^\circ$-nilpotent orbit in $\mathfrak{l} \cap V$, where $L \subset G$ is a connected group with Lie algebra $\mathfrak{l}$. [40]

The class corresponding to $(\mathfrak{l}, \mathcal{O})$ will be denoted by $J(\mathfrak{l}, \mathcal{O})$. We then have $J(\mathfrak{l}, \mathcal{O}) = M \cdot (\mathfrak{c}_V(\mathfrak{l})^\bullet + \mathcal{O})$, where $\mathfrak{c}_V(\mathfrak{l})$ is the centralizer of $\mathfrak{l}$ in V and $\mathfrak{a}^\bullet$ denotes the set of elements of $\mathfrak{a}$ with maximal dimension of M-orbit. It is fairly easy to show that the elements of a given decomposition class share a common M-orbit dimension. Moreover, we have

$$\operatorname{mod}(J(\mathfrak{l}, \mathcal{O})) = \dim \mathfrak{c}_V(\mathfrak{l}).$$

We say that $\mathcal{O}$ is *rigid* if it forms a single sheet for $((M \cap L)^\circ, \mathfrak{l} \cap V)$.

Ground results concerning sheets and decomposition classes can be stated as follows.

Theorem 4.2. *1. When (M, V) is the adjoint action associated with a reductive Lie algebra, [BK79, Bo81, IH05, Pe78]*

(a) Sheets are union of decomposition classes and the closure of a decomposition class is a union of decomposition classes.

[38] called Jordan classes there

[39] In the Lie algebra case this coincides with the classical notion of a Levi factor.
In the symmetric case, some types of Levi factors of $\mathfrak{g}$ might not appear as centralizers of elements of $\mathfrak{g}_1$.

[40] In the Lie algebra case, this is just a nilpotent L-orbit in $\mathfrak{l}$.
In the symmetric case, with obvious notation, this is an $(L_0)^\circ$-orbit in $\mathfrak{l}_1$.

(b) *Decomposition classes are finitely many. They are locally closed, irreducible, and smooth.*

(c) *Each sheet admits a unique dense decomposition class* [41]*, and a unique nilpotent decomposition class*[42]*.*

(d) *In a sheet, the inclusion relation on closures of decomposition classes is encoded by the* parabolic induction[43] *of nilpotent orbits:*

$$J(\mathfrak{l}_1, \mathcal{O}_1) \subset \overline{J(\mathfrak{l}_2, \mathcal{O}_2)}^{\bullet} \Leftrightarrow Ind_{\mathfrak{l}_2}^{\mathfrak{l}_1}(\mathcal{O}_2) = \mathcal{O}_1 \quad \textit{(up to conjugacy of pairs } (\mathfrak{l}_i, \mathcal{O}_i)\textit{)}.$$

(e) *Sheets are smooth in classical types. There exists a singular sheet in the exceptional type* G_2*.*

2. *When* (M, V) *is the action associated with a reductive symmetric Lie algebra*

(a) *1b) and 1c) (except unicity of the nilpotent decomposition class) remain valid, see [TY05] and references therein.*

(b) *Sheets in classical types A, B, C, D are smooth. They are studied in details in types AI, AII, AIII. [Bu11a] (PhD results)*

(c) *1a) remains valid over* $\mathbb{C}$[44]*. More precisely, closures of decomposition classes are unions of decomposition classes [Le11].*

The notion of *parabolic induction* mentioned in 1d) is crucial to study many aspect of sheets. For instance, it is used to

- parametrise sheets, [BK79]
- characterise sheets as the subvarieties of the form $\overline{J(\mathfrak{l}, \mathcal{O})}^{\bullet}$ with rigid orbit $\mathcal{O} \subset \mathfrak{l}$. [Bo81]
- show that sheets are stratified by decomposition classes. (cf. (1a)) [BK79, Bo81]
- compute rigid orbits algorithmically and provide combinatorial descriptions, for each sheet, of the decomposition classes $J(\mathfrak{l}, \mathcal{O})$ belonging to it. [Ke83, Sp80]
- ...

Unfortunately, it is hard to generalise this notion nicely in the symmetric setting. This can already be seen in the (SO_2, S_2)-case where the dense sheet S^{reg} contains both semisimple and nilpotent elements while there is no Borel subalgebra of $\mathfrak{g}$ containing elements of both types.

[41] with $\dim G \cdot x_s$ maximal, *i.e.* a decomposition class “as semisimple as possible”

[42] this class is then a single orbit

[43] see [Sp80, Bo81, TY05] for definitions and properties of parabolic induction, respectively *en français*, *auf Deutsch* and in english.

[44] using analytic methods

Following [Ka83] and [IH05] in the Lie algebra case, the study of [Bu11a] in the symmetric case shows that we can read much information about sheets on their so-called *Slodowy slices* [Sl80]. With Pascal Hivert (Univ. Versailles), the author developed in [BH16] a notion of *slice induction* which allows to recover most of the properties initially deduced from parabolic induction, as well as finer geometric properties of sheets.

More precisely, given a nilpotent element $e \in V$, we consider an affine subspace $e + U \subset V$ transversal to the orbit $M \cdot e$ at e. We do this by setting $U := \mathfrak{g}^f \cap V$ where (e, h, f) is a (normal[45]) $\mathfrak{sl}_2$-triple. Given a locally closed M-stable cone $S \subset V$, we can consider its Slodowy slice $e + X(S, e) := S \cap (e + U)$ which has the following properties

Lemma 4.3. *1.* $e \in \overline{S} \Leftrightarrow X(S, e) \neq \emptyset$

2. In this case, the orbit map

$$\begin{cases} M \times (e + X(S,e)) & \to & S \\ (m, x) & \mapsto & m \cdot x \end{cases}$$

is a smooth dominant morphism with relative dimension $\dim M \cdot e$.

The idea of (2) is that the action by M is transverse to the slice. The direct sense in (1) mainly follows from (2) applied to $\overline{S}$. The reciprocal follows from the existence of a 1-parameter subgroup of $M \times \mathbb{k}^*$ fixing e and contracting U to 0, where $\mathbb{k}^*$ acts by scalar multiplication.

Studying $X(S, e)$ is often easier than studying S. The first reason is that intersecting with the transversal space allows to "throw off the orbit part" by cutting off $\dim M \cdot e$ coordinates. Another reason is that if S is closed in $V^{(m)}$ with $m = \dim M \cdot e$, (thus not closed in V, unless $m = 0$) then $X(S, e)$ becomes a closed subvariety of U.

Following Lemma 4.3 and Theorem 4.2 (1d), we define the following.

Definition 4.4. Given $(\mathfrak{g}, \mathcal{O}_1)$ and $(\mathfrak{l}_2, \mathcal{O}_2)$, we say that $(\mathfrak{l}_2, \mathcal{O}_2)$ *slice induces* $(\mathfrak{l}_1, \mathcal{O}_1)$ when the following holds

- $X(J_2, e_1) \neq \emptyset$ where $J_2 := J(\mathfrak{l}_2, \mathcal{O}_2)$ and $e_1 \in \mathcal{O}_1$
- $\dim M \cdot \mathcal{O}_2 = \dim \mathcal{O}_1$.

This definition extends to pairs $(\mathfrak{l}_1, \mathcal{O}_1)$ and $(\mathfrak{l}_2, \mathcal{O}_2)$ with $\mathfrak{l}_2 \subset \mathfrak{l}_1$, by considering $\mathfrak{l}_1$ as the ambient algebra.

This definition is then sufficiently robust to get the following result.

[45]in the symmetric case, this condition means that $e, f \in \mathfrak{g}_1$, and thus $h \in \mathfrak{g}_0$. In the Lie algebra case, we only need an ordinary $\mathfrak{sl}_2$-triple.

Theorem 4.5. *[BH16] Assume that $(M, V) = (G_0, \mathfrak{g}_1)$ is the representation associated with a symmetric Lie algebra.*

Then the results (1a) and (1d) of Theorem 4.2 remain valid, replacing parabolic induction by slice induction.

Comments and perspectives

Theorem 4.5 also provides a new proof of these results in the setting of ordinary Lie algebras. Moreover, it shows that slice induction and parabolic induction coincide in this case.

In addition, it seems that the consequences listed below Theorem 4.2 can mostly be adapted in the symmetric setting using slice induction. The first three points are stated in [BH16]. For instance, each sheet can be parameterised as $S = \bigcup_i M.(e_i + X(S, e_i))$ where the e_i are representatives of the nilpotent orbits in S. The author plans to investigate other consequences in forthcoming studies.

Also, in yet unpublished works, the author describes an effective[46] algorithm computing $X(V^{(m)}, e)$ for any nilpotent element e with $\dim M \cdot e = m$. This allows to decide whether a given nilpotent orbit is rigid or not. Thanks to Lemma 4.3, this also allows to study whether the sheets are smooth or not, even in the Lie algebra case. In particular, examples of singular sheets were obtained in each exceptional (ordinary) Lie algebra (cf. Theorem 4.2 (1e))

Among the reasonable **goals to pursue in the symmetric setting**, one should then look for:

- The classification of sheets in each symmetric Lie algebra.
- The classification of the decomposition classes lying in each sheet.
- Complete induction tables in order to classify the pairs $(\mathfrak{l}_1, \mathcal{O}_1)$, $(\mathfrak{l}_2, \mathcal{O}_2)$ for which $J(\mathfrak{l}_1, \mathcal{O}_1) \subset \overline{J(\mathfrak{l}_2, \mathcal{O}_2)}^{(\bullet)}$.

In order to fully answer to the first two points, one of the main remaining difficulties is the conjugacy problem: "In a given symmetric Lie algebra, decide whether isomorphic pairs $(\mathfrak{l}, \mathcal{O}_1)$, $(\mathfrak{l}, \mathcal{O}_2)$ are conjugate or not".

In order to get complete induction tables, one should compute whether $X(J(\mathfrak{l}, \mathcal{O}), e) \neq \emptyset$ [47] for each $(\mathfrak{l}, \mathcal{O})$ and nilpotent orbit $M \cdot e$. For non-rigid $\mathcal{O}$ the computation of $X(J(\mathfrak{l}, \mathcal{O}), e)$ turns out to be more involved than for rigid orbits.

Upon finishing this dissertation, the article [CES20] was prepublished. It generalises the results of [BH16] to the θ-groups setting. The general strategy is similar to [BH16], making use of a recent study [Po18] of decomposition classes in this setting.

[46] at least in the low rank cases, including all the exceptional ones

[47] equivalently, whether $X(\overline{J(\mathfrak{l}, \mathcal{O})}, e) \neq \emptyset$, thanks to Lemma 4.3 (1)

5 Symplectic reductions

As explained in Section 1.4, the commuting scheme fits in the broader class of schemes of the form $\mathcal{C}(G, V) := \mu^{-1}(0) \subset V \times V^*$ with μ the moment map associated with an arbitrary representation (G, V), see (1.10).

In light of Theorem 2.1 (and Theorem 5.4), it is interesting to look at representations for which the notion of Cartan subspace still makes sense. This is the case for representations arising from reductive θ-groups and for polar representations.

While sharing many similarities with the Lie algebra case, these more general settings allow new phenomenons, including counter-examples to open conjectures or theorems which are valid for reductive Lie algebras. In this direction, the author showed the following result:

Theorem 5.1. *[Bu18] Assume that (G, V) is the representation associated with a simple θ-group and assume also that (G, V) is stable and locally free. Then $\mathcal{C}(G, V)$ is normal in all but* 5 *cases.*

Remark 5.2. The representations associated with θ-groups are always visible. Then the quite restrictive assumptions of stability and local freeness imply by (1.7) and Theorem 2.6 (5) that normality is only a matter of codimension of the singular locus. Thus Theorem 5.1 can be seen as an attempt to generalise Theorem 2.6 (1) (and Theorem 3.1) in very favorable cases.

The main lesson is that even in these favorable cases, there are counter-examples to normality. All these 5 counter-examples arise in *exceptional* θ-groups of rank 1.

As indicated in Section 1.4, we can always define the categorical quotient of an affine variety under the action of a reductive group. The quotient $V/\!\!/G$ is a variety whose points are in bijection with closed G-orbits in V. The following result is a generalization of Chevalley's restriction theorem for reductive Lie algebras. It was previously obtained in [KR71] (resp. [Vi76]) for actions associated with symmetric Lie algebras (resp. θ-groups).

Theorem 5.3. *[DK85] Let (G, V) be a polar representation. Then $V/\!\!/G$ is isomorphic to $\mathfrak{c}/\!\!/W (= \mathfrak{c}/W)$ where $\mathfrak{c}$ is a Cartan subspace of V and W is the associated* Weyl group[48]*.*
Since W is a finite group generated by reflections, these quotient spaces are isomorphic to affine spaces.

In general, the quotient of an affine space by a finite group is not isomorphic to an affine space. It is however a variety with nice properties[49], which is called an *orbifold*

[48] *i.e.* $W = N_G(\mathfrak{c})/Z_G(\mathfrak{c})$ where $N_G(\mathfrak{c})$ and $Z_G(\mathfrak{c})$ respectively denote the stabilizer and the pointwise stabilizer of $\mathfrak{c}$ in G

[49] *e.g.* it is normal

In the setting of symplectic reduction, we consider the quotient scheme $\mu^{-1}(0)/\!\!/G = \mathcal{C}(G,V)/\!\!/G$. Then the following theorem is often considered as a "doubled version" of Chevalley's restriction theorem[50]. It was first proved by A. Joseph in [Jo97] in the setting of Lie algebras.

Theorem 5.4. *[Te00] If (G,V) is the representation associated with a symmetric Lie algebra, then the quotient variety $\mathcal{C}(G,V)_{red}/\!\!/G$ is isomorphic to $(\mathfrak{c}\times\mathfrak{c})/W$. In particular, it is normal.*

Note that when X is normal, then $X/\!\!/G$ is automatically normal. This theorem can thus be seen as a good test for the validity of Conjecture 2.2. On the other hand, we have seen that this normality conjecture fails in more general settings:

- In the symmetric setting, $\mathcal{C}(G,V)$ might be reducible, cf. Theorem 2.6 (3). In these cases, the irreducible components meet at 0 so the $\mathcal{C}(G,V)_{red}$ cannot be normal. The above result then shows that the quotient is better behaved than the original variety.
- In the θ-group setting, even if $\mathcal{C}(G,V)$ is reduced and (S_2), normality can fail, cf. Theorem 5.1.

Another important seminal result in this direction deals with the normality of *Nakajima quiver varieties*[51]. The following is a simplified version of this result.

Theorem 5.5. *[CB03] If (G,V) is a representation arising from a quiver setting (see Section 1.3), then $\mathcal{C}(G,V)_{red}/\!\!/G$ is a normal variety*

With Christian Lehn (Univ. Chemnitz, GER), Manfred Lehn (Univ. Mainz, GER) and Ronan Terpereau (Univ. Bourgogne), we got the following generalisation of Theorem 5.4

Theorem 5.6. *[BLLT17] Let (G,V) be a polar representation.*

1. *There is a canonical scheme morphism $r : (\mathfrak{c}\times\mathfrak{c}^*)/W \to \mathcal{C}(G,V)/\!\!/G$ which is also a Poisson morphism.*
2. *If, in addition, (G,V) is locally free, visible and stable, then r is an isomorphism. In particular, $\mathcal{C}(G,V)/\!\!/G$ is an orbifold, so is normal*[52].

Moreover, dropping some hypothesis on (G,V), we have exhibited several counter-examples to the second point of the theorem. More precisely

[50] or of Theorem 5.3 for generalizations

[51] *i.e.* a symplectic reduction associated to a quiver representation space, possibly after adding *framings*, also considering fibers at points other than 0 and where the quotient is taken with respect to some possibly non-trivial stability parameter

[52] Note that irreducibility and reducedness of $\mathcal{C}(G,V)/\!\!/G$ already follow from Theorem 2.6 (5)

Example 5.7.

- There are some polar representations for which r is not bijective. In some of these cases, $\mathcal{C}(G,V)_{red}/\!\!/G$ is even not irreducible (thus not normal).
- There exists a symmetric Lie algebra for which $\mathcal{C}(G,V)/\!\!/G$ is not reduced. In particular, $\mathcal{C}(G,V)$ is not reduced in this case.

We found the counter-examples of the first kind only in non-visible cases. On the other hand, in order to avoid the difficulty raised by the second counter-example, we can consider the reduced morphism r_{red} [53] yielded by the scheme morphism r of Theorem 5.6. We then stated the following reasonable conjecture:

Conjecture 5.8. *If (G,V) is polar and visible, then r_{red} is an isomorphism. In particular, $\mathcal{C}(G,V)_{red}/\!\!/G$ is normal.*

In support of this conjecture, the author also studied arbitrary representations of tori. Tori simply are the multiplicative groups of the form $T_k = (\Bbbk^\times)^d$. Their representation theory is much simpler than that of general reductive groups. In this setting, the following result was shown.

Theorem 5.9. *[Bu18] Assume that (T,V) is a representation with T a torus.*

1. *$\mathcal{C}(T,V)$ is normal.*
2. *If the representation is not visible, then there exists some closed orbits in $\mathcal{C}(T,V)$ of the form $T.(x,y)$ with $T.x$ and $T.y$ non-closed. In particular, no natural analogue of the map r can be bijective.*
3. *If the representation is visible, then it is polar and Conjecture 5.8 holds for (T,V).*

Comments and perspectives

In many interesting cases, it is not hard to show that r_{red} is a bijection. For instance, this is the case for the representation associated to an arbitrary θ-group. In this case, the isomorphism statement in Conjecture 5.8 is equivalent to the normality of $\mathcal{C}(G,V)_{red}/\!\!/G$.

Deciding whether $\mathcal{C}(G,V)_{red}/\!\!/G$ is normal for any given representation is a classical question. The answer is not always positive (see Example 5.7). However in [HSS20], it is shown that the normality holds for any "sufficiently large" representation of a group G. For these representations, there is no hope of finding something like a Cartan subspace which would help interpreting $\mathcal{C}(G,V)_{red}/\!\!/G$ as an orbifold. In comparison, the visible polar representations studied here are quite "small".

In addition to Conjecture 5.8, we can mention some interesting **open problems**:

[53] with reduced target $\mu^{-1}(0))_{red}/\!\!/G$

- Describe the quotient in "intermediate cases", where there is no Cartan subspace, but where the techniques of [HSS20] do not apply.

- In the cases covered by Conjecture 5.8, reconstruct the invariants $\Bbbk[\mathcal{C}(G,V)_{(red)}]^G$ from the invariants $\Bbbk[\mathfrak{c}]^W \cong \Bbbk[V]^G$ and $\Bbbk[\mathfrak{c}^*]^W \cong \Bbbk[V^*]^G$. For instance, it has already been shown in [Hu97], that $\Bbbk[\mathcal{C}(\mathfrak{g})_{red}]^G$ is generated by the *polarisations* (resp. *generalized polarisations*) of elements of $\Bbbk[\mathfrak{g}]^G$ in the classical cases of type A, B, C (resp. D).

6 Hilbert schemes via commuting varieties and quivers

6.1 Hilbert schemes

Central objects of interest in this section are Hilbert schemes. They parametrise certain subschemes (here, 0-dimensional ones) of a given variety (here, $\Bbbk^2$, with $\Bbbk$ algebraically closed). We can partition the set of 0-dimensional subschemes via their *length*. If I_z is the ideal in $\Bbbk[X,Y]$ (the algebra of polynomial function on $\Bbbk^2$) generated by the equations defining a subscheme $z \subset \Bbbk^2$, then the length of z is defined as

$$\operatorname{length}(z) := \operatorname{codim}_{\Bbbk[X,Y]} I_z.$$

It is finite precisely when z is 0-dimensional. For instance, a reduced subscheme with n distinct points has length n. Examples of non-reduced subschemes of length 2 supported on the single point $(0,0)$ are given in (1.6). They are said to be *curvilinear* since each is a subscheme of a smooth curve of $\Bbbk^2$. Given $n \in \mathbb{N}^*$, a smooth curve $C \subset \Bbbk^2$ and a point $(x,y) \in C$, there is exactly one (necessarily curvilinear) subscheme of C of length n supported on (x,y). Some examples of subschemes of length 3 are given below:

$$x^3 = 0 = y - ax - bx^2 \quad (a,b \in \Bbbk) \qquad \text{the length 3 curvilinear subscheme at } (0,0) \text{ of the parabola } y = ax + bx^2 \tag{6.1}$$

$$\{x^2 = xy = y^2 = 0\} \qquad \text{the length 3 \textit{fat point} supported on } (0,0) \tag{6.2}$$

We can define the (punctual) Hilbert scheme (of length n in the affine plane) set-theoretically via:

$$\operatorname{Hilb}^n = \operatorname{Hilb}^n(\Bbbk^2) := \{\text{subschemes } z \subset \Bbbk^2 |\ \operatorname{length}(z) = n\} \qquad (n \in \mathbb{N})$$

We refer to [EH00, Be12] for a definition in terms of representable functors which allows to define a natural scheme structure [54] on the above set[55]. For instance, in Hilb^2, the subscheme defined in (1.6) is in the closure of the set of reduced subschemes of the form $\{(t,at),(-t,-at)\}$ $(t \in \Bbbk^\times)$.

Along the same ideas we can construct many interesting variations of such Hilbert schemes. For instance, the following scheme parametrises the "least possibly reduced" subschemes:

$$\operatorname{Hilb}^n_0 = \operatorname{Hilb}^n_0(\Bbbk^2) := \{z \in \operatorname{Hilb}^n \mid z \text{ is supported on } (0,0)\}$$

[54] separated of finite type over $\Bbbk$

[55] and also on $\operatorname{Hilb}^n(X)$, for any scheme X

Scheme-theoretically, it is the fiber at 0 of the Hilbert-Chow morphism[56] [Be12]. Moreover, in the study of these Hilbert schemes we are naturally led to consider *nested Hilbert schemes* [Ch98][57].

$$\mathrm{Hilb}^{\boldsymbol{n}}(\Bbbk^2) := \left\{ (z_1, \ldots, z_s) \,\middle|\, \begin{array}{c} \forall i\ z_i \in \mathrm{Hilb}^{n_i} \\ \forall i\ z_i \subset z_{i+1} \end{array} \right\} \qquad \left(\begin{array}{c} \underline{\boldsymbol{n}} = (n_1, \ldots, n_s) \in \mathbb{N}^s \\ n_1 \leqslant \cdots \leqslant n_s = n \end{array} \right)$$

$$\mathrm{Hilb}_0^{\boldsymbol{n}}(\Bbbk^2) := \left\{ (z_1, \ldots, z_s) \,\middle|\, \begin{array}{c} \forall i\ z_i \in \mathrm{Hilb}_0^{n_i} \\ \forall i\ z_i \subset z_{i+1} \end{array} \right\} \qquad \left(\begin{array}{c} \underline{\boldsymbol{n}} = (n_1, \ldots, n_s) \in \mathbb{N}^s \\ n_1 \leqslant \cdots \leqslant n_s = n \end{array} \right)$$

For instance, Hilb_0^2 is isomorphic to the projective line $\mathbb{P}^1_\Bbbk$. Heuristically, the points of Hilb_0^2 are in bijection with the lines of $\Bbbk^2$ passing through the origin: with each line we can associate its length 2 curvilinear subscheme supported at the origin, see (1.6). Another geometric example: in Hilb_0^3, the fat point (6.2) lies in the closure of the set of the curvilinear schemes of the form (6.1) [58].

The simplest points on a given Hilbert scheme, are the curvilinear ones. The set of such points forms an irreducible subvariety in each above-defined Hilbert scheme and the closure turns out to be an irreducible component. We call this component the *curvilinear component* of the considered Hilbert scheme. This should be thought of as an analogue of the principal component for commuting varieties. For $Hilb^n$ the general[59] curvilinear points are unions of n reduced points. If $z_n \in \mathrm{Hilb}_0^n$ is curvilinear, for $k \in [\![0, n]\!]$, there exists a unique $z_k \in \mathrm{Hilb}_0^k$ such that $z_k \subset z_n$. Thereby, we get a bijective morphism from the curvilinear locus of $\mathrm{Hilb}_0^{\boldsymbol{n}}$ to that of Hilb_0^n.

We now state a few results of importance concerning the geometry of Hilbert schemes:

Theorem 6.1. *Under the above notation,*

1. *The scheme* $\mathrm{Hilb}^n(\Bbbk^2)$ *is smooth, irreducible of dimension* $2n$ *[Fo68].*
   The curvilinear component of $\mathrm{Hilb}^{\boldsymbol{n}}(\Bbbk^2)$ *is also of dimension* $2n$.

2. $\mathrm{Hilb}_0^n(\Bbbk^2)$ *is irreducible of dimension* $n-1$ *[Br77].*
   The curvilinear component of $\mathrm{Hilb}_0^{\boldsymbol{n}}(\Bbbk^2)$ *is also of dimension* $n-1$.

3. $\mathrm{Hilb}^{\boldsymbol{n}}$ *and* $\mathrm{Hilb}_0^{\boldsymbol{n}}$ *are connected. The scheme* $\mathrm{Hilb}^{k,n}$ *is smooth if and only if* $k \in \{0, 1, n-1, n\}$[60] *[Ch98]. Moreover* $\mathrm{Hilb}_0^{n-1,n}$ *is irreducible [CE15]*

Note that it follows from items 1 and 2 that general points of Hilb^n and Hilb_0^n are curvilinear.

[56] a natural morphism sending a subscheme of length n of $\Bbbk^2$ to its support in $(\Bbbk^2)^n/\mathfrak{S}_n$

[57] see [BFT20] for some motivations coming from physics

[58] Clue: set $a_t := t$ and $b_t := t^{-1}$ so that the equation $y - a_t x - b_t x^2 = 0$ yields $x^2 = -t^2 x + ty$ and $xy = (tx^2 + t^{-1}x^3 =)tx^2$ and consider $t \to 0$.

[59] *i.e.* for points in some well-chosen dense open subset of the set of curvilinear points

[60] and, essentially, these are the only smooth cases of $\mathrm{Hilb}^{\boldsymbol{n}}$. For instance, $\mathrm{Hilb}^{1,2,3}$ is smooth, but it is isomorphic to $\mathrm{Hilb}^{2,3}$

6.2 Link with commuting varieties in type A

In [Na99], Nakajima built a connection between the Hilbert scheme $\mathrm{Hilb}^n(\Bbbk^2)$ and the type A commuting scheme $\mathcal{C}(\mathfrak{gl}_n)$. Such result allows to reduce the study of Hilbert schemes to that of commuting varieties, which are more "linear" objects.

More precisely, given $(x,y) \in \mathcal{C}(\mathfrak{gl}_n)$ and $v \in V := \Bbbk^n$ a *cyclic element*[61] for the commuting pair (x,y), we define the following ideal of codimension n of $\Bbbk[X,Y]$:

$$\ker\left(\varphi : \begin{cases} \Bbbk[X,Y] & \to & V \\ P & \mapsto & P(x,y)v \end{cases}\right) =: I_{(x,y),v}.$$

This yields a map

$$\tilde{\pi} : \begin{cases} \tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n) & \to & \mathrm{Hilb}^n \\ ((x,y),v) & \mapsto & I_{(x,y),v} \end{cases} \tag{6.3}$$

where $\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n) := \{((x,y),v) \in \mathcal{C}(\mathfrak{gl}_n) \times V \,|\, v \text{ is cyclic for } (x,y)\}$ is a GL_n-stable[62] open subset of $\mathcal{C}(\mathfrak{gl}_n) \times V$.

Theorem 6.2. *[Na99] The map $\tilde{\pi}$ is a geometric quotient morphism*[63]*, thus* $\mathrm{Hilb}^n \cong \tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)/GL_n$.

In fact, the action of GL_n on $\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)$ is free. Moreover, if x, y, v, v' are such that $((x,y),v), ((x,y),v') \in \tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)$, then these two tuples lie in the same GL_n-orbit and we have $I_{(x,y),v} = I_{(x,y),v'} = \{P \in \Bbbk[X,Y] | P(x,y) = 0\}$. We denote by $I_{x,y}$ this last set. Thus $\tilde{\pi}$ induces a set-theoretic quotient map by GL_n

$$\pi : \begin{cases} \mathcal{C}^{cyc}(\mathfrak{gl}_n) & \to & \mathrm{Hilb}^n \\ (x,y) & \mapsto & I_{x,y} \end{cases} \tag{6.4}$$

where $\mathcal{C}^{cyc}(\mathfrak{gl}_n) := \{(x,y) \in \mathcal{C}(\mathfrak{gl}_n) \text{ which admits a cyclic element}\}$.

It is unclear whether π is a morphism, nevertheless the geometries of $\mathcal{C}^{cyc}(\mathfrak{gl}_n)$ and Hilb^n are intimately linked by Theorem 6.2. For instance, we can recover Theorem 6.1 (1) from results on the commuting variety. Indeed, Theorem 2.1 implies that $\mathcal{C}^{cyc}(\mathfrak{gl}_n)$ and thus $\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)$ and $\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)/GL_n$ are irreducible. Since the GL_n-action on $\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)$ is free, the smoothness statement follows from that of $\mathcal{C}^{cyc}(\mathfrak{gl}_n) \subset \mathcal{C}(\mathfrak{gl}_n) \setminus \mathcal{C}^{irr}(\mathfrak{gl}_n)$, see (3.1).

The above constructions have analogues for the other Hilbert schemes. Let $\mathfrak{p}_{\boldsymbol{n}} := \{x \in \mathfrak{gl}_n | \forall i,\, x(V_{n_i}) \subset V_{n_i}\}$ and $P_{\boldsymbol{n}} := \mathfrak{p}_{\boldsymbol{n}} \cap GL_n$, where $\{0\} =: V_{n_s} \subset V_{n_{s-1}} \subset \cdots \subset V_{n_1} \subset V$ is a fixed partial flag of V with $\dim V/V_{n_i} = n_i$ for each i. They are respectively parabolic subalgebras and parabolic subgroups of GL_{n_s}. Setting $n_0 := 0$, the associated Levi have blocks of size $n_i - n_{i-1}$.

[61] *i.e.* as a $\Bbbk[x,y]$-module, V is generated by v. In other words $\langle x^k y^l v\rangle_{k,l\in\mathbb{N}} = V$.

[62] the action of GL_n is given by $g \cdot ((x,y),v) = (gxg^{-1}, gyg^{-1}, gv)$

[63] Here $\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n)$ is non-affine in general, but we can find a cover by GL_n-stable affine open subvarieties whose quotient maps can be glued together

Hilb	Hilb^n	Hilb_0^n	$\mathrm{Hilb}^{\underline{\boldsymbol{n}}}$	$\mathrm{Hilb}_0^{\underline{\boldsymbol{n}}}$
$\mathcal{C}$	$\mathcal{C}(\mathfrak{gl}_n)$	$\mathcal{C}^{nil}(\mathfrak{gl}_n)$	$\mathcal{C}(\mathfrak{p}_{\underline{\boldsymbol{n}}})$	$\mathcal{C}^{nil}(\mathfrak{p}_{\underline{\boldsymbol{n}}})$
G	GL_n	GL_n	$P_{\underline{\boldsymbol{n}}}$	$P_{\underline{\boldsymbol{n}}}$
reference	[Na99]	[Ba01]	[BE16][65]	

Table 2: Hilbert schemes and commuting varieties

Proposition 6.3. *[Na99, Ba01, BE16] Let $\mathcal{C}$,* Hilb *and G be as in Table 2. Let $\mathcal{C}^{cyc} := \mathcal{C} \cap \mathcal{C}^{cyc}(\mathfrak{gl}_n)$ and $\tilde{\mathcal{C}}^{cyc} := \tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n) \cap (\mathcal{C} \times V)$.*

Then,

- *The group G acts on $\tilde{\mathcal{C}}^{cyc}$ and $\mathcal{C}^{cyc}$. The action on $\tilde{\mathcal{C}}^{cyc}$ is free.*
- *The restriction $\tilde{\pi}_{|\tilde{\mathcal{C}}^{cyc}}$ is a geometric quotient by G. Thus* Hilb $\cong \tilde{\mathcal{C}}^{cyc}/G$*.*
- *The restriction $\pi_{|\mathcal{C}^{cyc}}$ is a set theoretic quotient by G.*
  It induces a bijection between
  - *the set of irreducible components of* Hilb *of dimension m*
  - *the set of irreducible components of $\mathcal{C}^{cyc}$ of dimension $m + \dim G - n$.*[64]

This correspondence is useful in both directions. Indeed, V. Baranovsky used in [Ba01] the irreducibility of Hilb_0^n (see, Theorem 6.1 (2)) to prove the irreducibility of $\mathcal{C}^{nil}(\mathfrak{gl}_n)$. Then, in [Pr03] (see Theorem 2.6 (2)), A. Premet gave a proof of the irreducibility of $\mathcal{C}^{nil}(\mathfrak{gl}_n)$ valid over an algebraically closed field of arbitrary characteristic, thus proving the irreducibility of Hilb_0^n in this setting.

With Laurent Evain (Univ. Angers), we then got the following more specific results by making use of the above correspondence.

Theorem 6.4. *[BE16]*

1. *$\mathcal{C}^{nil}(\mathfrak{p}_{k,n})$ and $\mathrm{Hilb}_0^{k,n}(\mathbb{k}^2)$ are irreducible if and only if $k \in \{0, 1, n-1, n\}$.*
2. *Assume that $n \geqslant 4$ and $k \in \{2, n-2\}$. Then $\mathcal{C}^{nil}(\mathfrak{p}_{k,n})$ and $\mathrm{Hilb}_0^{k,n}(\mathbb{k}^2)$ are equidimensional of respective dimension $\dim \mathfrak{p}_{k,n} - 1$ and $n-1$. They both have $\left\lfloor \frac{n}{2} \right\rfloor$ irreducible components.*

[64] alternatively, this is the set of irreducible components of $\mathcal{C}$ of dimension $m + \dim G - n$ which meet $\mathcal{C}^{cyc}$. Notably, the principal component of $\mathcal{C}$ meets $\mathcal{C}^{cyc}$ and corresponds to the curvilinear component of Hilb via this bijection

[65] In this paper, the results were only stated for the last column, but the proofs work in the same way for the 3rd column

It is striking in these examples that $\mathrm{Hilb}_{(0)}^{k,n}$ and $\mathrm{Hilb}_{(0)}^{n-k,n}$ share similar properties. To some extent, this can be explained by the fact that they are related to the isomorphic Lie algebras $\mathfrak{p}_{k,n}$ and $\mathfrak{p}_{n-k,n}$ respectively. However the resulting GIT quotients of $\mathcal{C}(\mathfrak{p}_{\underline{n}})$ are constructed with respect to different stability conditions.

6.3 Modality of $\mathfrak{p}^{nil}$ and consequences

From Proposition 6.3, we are led to study the (nilpotent) commuting varieties of any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{gl}_n$. From Remark 2.5, and Theorem 2.6 (6), it is enlightening to look at the modality of the action of P on $\mathfrak{p}$ and $\mathfrak{p}^{nil}$, as defined below (2.5).

With Magdalena Boos (Univ. Bochum, GER), we tackled the classification of parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{gl}_n$ satisfying $\mathrm{mod}(P, \mathfrak{p}^{nil}) = 0$, that is such that $\mathfrak{p}^{nil}$ has a finite number of P-orbits. Similar questions had already been studied in various contexts[66]. We can cite

Theorem 6.5. *1. $P_{k,n}$ acts on $\mathfrak{p}_{k,n}^{nil}$ with finitely many orbits if and only if $k \leqslant 5$ or $k \geqslant n-5$. [Mu00]*

2. $P_{\underline{n}}$ acts on the nilradical of $\mathfrak{p}_{\underline{n}}$ with finitely many orbits if and only if the corresponding Levi has at most 5 blocks [HR99]

Our main results can be stated as follows.

Theorem 6.6. *[BB19] Let $\underline{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$ with $n_1 < \dots < n_s$. Let $d_{\underline{n}} := (n_s - n_{s-1}, \dots, n_2 - n_1, n_1) \in \mathbb{N}^s$ be the block sizes of a Levi of the parabolic subalgebra of $\mathfrak{p}_{\underline{n}} \subset \mathfrak{gl}_n$, then*

1. *$P_{\underline{n}}$ acts on $\mathfrak{p}_{\underline{n}}^{nil}$ with finitely many orbits if and only if $d_{\underline{n}}$ appears (up to symmetry) in Diagram 2.*
2. *In these cases, $\dim \mathcal{C}^{nil}(\mathfrak{p}_{\underline{n}}) = \dim \mathfrak{p}_{\underline{n}} - 1$ and $\dim \mathrm{Hilb}_0^{\underline{n}} = n-1$.*
3. *If $6 \leqslant k \leqslant n-6$, then $\dim \mathcal{C}^{nil}(\mathfrak{p}_{k,n}) \geqslant \dim \mathfrak{p}_{\underline{n}}$ and $\dim \mathrm{Hilb}_0^{k,n} \geqslant n$.*
4. *$\mathcal{C}(\mathfrak{p}_{200,400})$ is reducible.*

Remark 6.7.

- The infinite cases[67] are displayed in Diagram 1.
- Statement (1) is in fact valid over an arbitrary infinite field.
- In order to prove statement (2), we only have to apply Remark 2.5 and Proposition 6.3. The "-1" in these formulas comes from the fact that there is no distinguished orbit in $\mathfrak{p}_{\underline{n}}^{nil}$, but there are some in $\mathfrak{p}_{\underline{n}}^{nil} \cap \mathfrak{sl}_n$.

[66] An early appearance of the study of commuting varieties in the 0-modality case can be found in [Py75]

[67] *i.e.* the complementary cases to statement (1)

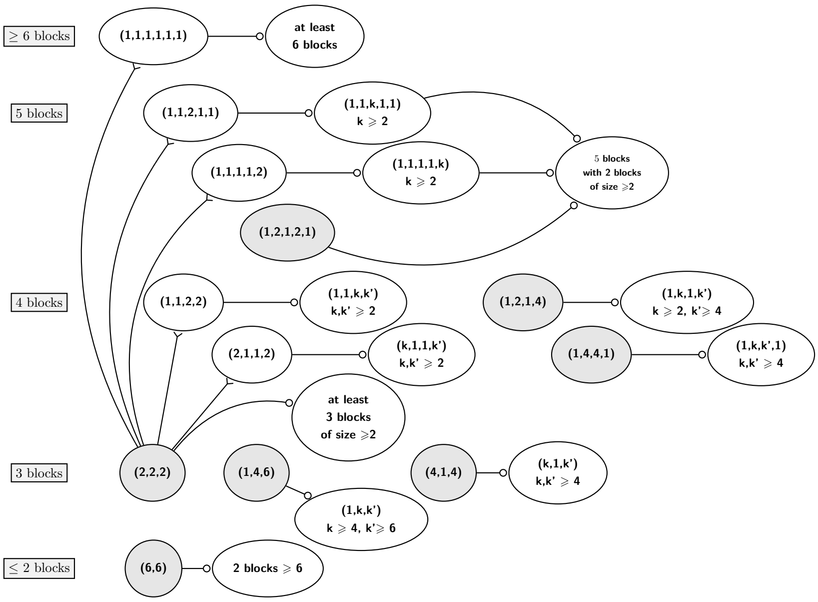


Figure 1: Infinite cases

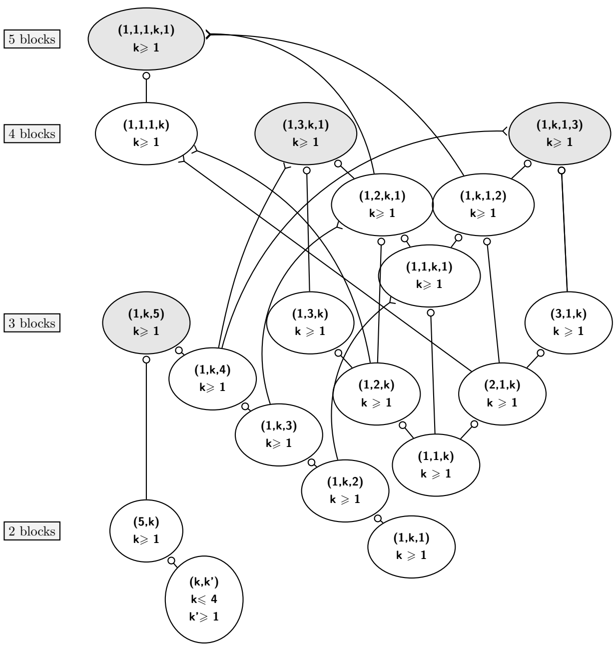


Figure 2: finite cases

- The first part of statement (3) is about maximal parabolics of $\mathfrak{gl}_n$.
  In order to prove it we need to show that, when an infinite family of orbits arise in this setting, then we can always manage to get an infinite family of distinguished orbits. This was done by relating distinguished elements of $\mathfrak{p}_{\underline{\boldsymbol{n}}}^{nil}$ to irreducible representations of the quiver $\mathcal{Q}_s$ defined below.
  In order to push the result down to the Hilbert scheme $\mathrm{Hilb}_0^{k,n}$, we had to show that for x lying in such an infinite family, the general pairs $(x, y) \in \mathcal{C}^{nil}(\mathfrak{p}_{k,n})$ have cyclic elements.

- In [GG18], the authors pointed out several examples of parabolics with reducible commuting variety. In type A, these examples were built using infinite families provided by Theorem 6.5 (2) and the resulting parabolics had Levi factors with at least 6 blocks. Statement (4) shows that maximal parabolics can also have a reducible commuting variety.

- The proof of the "if" part in (1) is done case by case on the darkened cases of Diagram 2. Then some reduction lemmas allow to deduce finiteness in the other cases.

6.4 Elements of proof via quiver representations

The aim of this subsection is to expose the key concepts behind the proof of Theorem 6.6 (1)-"only if part" and (4).

Following [HR99] and [BH00], we reinterpreted the classification of orbits of $\mathfrak{p}^{nil}$ in terms of quiver representations where one can use richly developed theories[68]. For $\underline{\boldsymbol{n}} = (n_1, \dots, n_s)$, we let $\mathcal{Q}_s$ be the quiver with s vertices, one loop β_i at each vertex ($i \in [\![1, s]\!]$) and one arrow α_i from vertex i to vertex $i+1$ ($i \in [\![1, s-1]\!]$). That is

$$\mathcal{Q}_s : \quad \underset{1}{\overset{\beta_1 \circlearrowleft}{\bullet}} \xrightarrow{\alpha_1} \underset{2}{\overset{\beta_2 \circlearrowleft}{\bullet}} \xrightarrow{\alpha_2} \underset{3}{\overset{\beta_3 \circlearrowleft}{\bullet}} \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{s-3}} \underset{s-2}{\overset{\beta_{s-2} \circlearrowleft}{\bullet}} \xrightarrow{\alpha_{s-2}} \underset{s-1}{\overset{\beta_{s-1} \circlearrowleft}{\bullet}} \xrightarrow{\alpha_{s-1}} \underset{s}{\overset{\beta_s \circlearrowleft}{\bullet}}$$

Then,

Proposition 6.8. *[BB19] There is a bijection*[69]

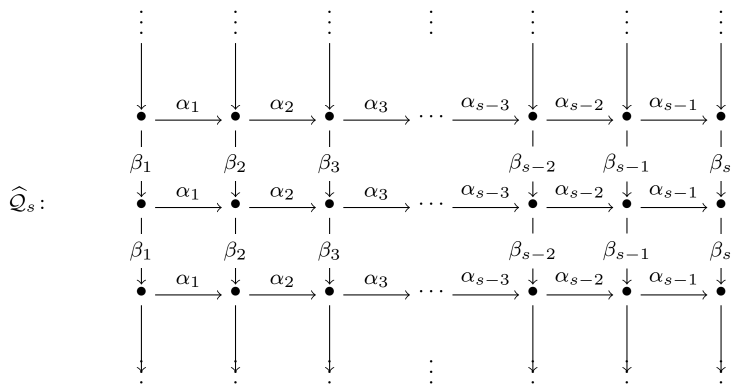
$$\left\{ P_{\underline{\boldsymbol{n}}}\text{-orbits in } \mathfrak{p}_{\underline{\boldsymbol{n}}}^{nil} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} GL_{\underline{\boldsymbol{n}}}\text{-orbits of representations of } \mathcal{Q}_s \\ \text{with dimension vector } \underline{\boldsymbol{n}}, \\ \text{satisfying the relations} \\ \beta_{i+1}\alpha_i = \alpha_i\beta_i, \text{ each } \beta_i \text{ is nilpotent,} \\ \text{and each } \alpha_i \text{ is injective.} \end{array} \right\}$$

[68] *e.g.* the BHV-list (from works of **B**ongartz, **H**appel and **V**ossieck) available in [GR92, §10] allows to classify all the quivers with relations into finite and infinite representation type.
[69] see [BB19, Lemma 3.1] for a more geometric statement

A bit more explicitly, given (the $GL_{\underline{n}}$-orbit of) a representation of $\mathcal{Q}_s$ satisfying the above requirements,

- the map provided by β_s defines an endomorphism of $V = V_{n_s}$
- the system of injective maps $(\alpha_i)_{i \in [\![1,s-1]\!]}$ defines a flag $V_{n_1} \subset \cdots \subset V_{n_s}$ (with $\dim V_{n_i} = n_i$) stabilised by β_s such that the associated parabolic is isomorphic to $\mathfrak{p}_{\underline{n}}$[70].

The next step is to consider the covering quiver $\widehat{\mathcal{Q}}_s$ of $\mathcal{Q}_s$ with respect to the given relations. This has the noteworthy effect of getting rid of the loops. We refer to [BG81] for covering theory.



We are interested in representations of $\widehat{\mathcal{Q}}_s$

- whose dimension vectors $(n_{i,j})_{i \in \mathbb{N}, j \in [\![1,s]\!]}$ satisfy $\sum_i n_{i,j} = n_j$
- such that the squares commute, *i.e.* $\beta_{i+1}\alpha_i = \alpha_i\beta_i$.
- with injective α_i's[71]

Covering theory then provides a map $\mathcal{F}$ from this variety of representations of $\widehat{\mathcal{Q}}_s$ to the variety of representations of $\mathcal{Q}_s$ involved in Proposition 6.8. Among other properties of this map, we mention

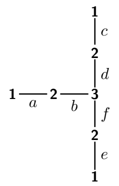
Proposition 6.9. *If M, M' are two representations of $\widehat{\mathcal{Q}}_s$ such that M is not isomorphic to a $\mathbb{Z}$-translate of M', then $GL_{\underline{n}} \cdot \mathcal{F}(M) \neq GL_{\underline{n}} \cdot \mathcal{F}(M')$.*

Thanks to Propositions 6.9 and 6.8, from infinite families of non-isomorphic representations of $\widehat{\mathcal{Q}}_s$ we get infinite families of $P_{\underline{n}}$-orbits in $\mathfrak{p}_{\underline{n}}^{nil}$. This turns out to be a powerful tool for the "only-if part" of Theorem 6.6 (1). For instance, we

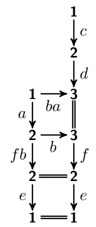
[70] The flag provided here has a embedding structure dual to the one defining $\mathfrak{p}_{\underline{n}}$. Still, the associated parabolic Lie algebras are isomorphic

[71] the nilpotency relation $\underbrace{\beta_i \circ \cdots \circ \beta_i}_{n} = 0$ is automatic since we look at representations with finite support

can fold the infinite families provided by the extended Dynkin quivers[72] in order to construct infinite families for $\widehat{\mathcal{Q}}_2$ satisfying the above-described conditions (injectivity, nilpotency and commuting square conditions).



Infinite family of representations for the extended Dynkin diagram in type E_6



Infinite family for $\widehat{\mathcal{Q}}_2$ with $\underline{\boldsymbol{n}} = (6, 12)$ *i.e.* $d_{\underline{\boldsymbol{n}}} = (6, 6)$

Lastly, in order to get Theorem 6.6 (4), we looked for (k, n) such that $\mathcal{C}(\mathfrak{p}_{k,n})$ has an irreducible component of dimension bigger than that of the principal component. More precisely, the modality of the principal component is equal to $n = n_s$. On the other hand, quiver theory implies that modality grows quadratically with respect to the dimension vector. We could push this fact through Propositions 6.9 and 6.8 to exhibit maximal[73] parabolics with $\mathrm{mod}(P_{k,n}, \mathfrak{p}_{k,n})(\geqslant \mathrm{mod}(P_{k,n}, \mathfrak{p}_{k,n}^{nil})) \geqslant n$.

6.5 Comments and perspectives

The above reasoning consists in firstly linking Hilbert schemes with commuting varieties, and then studying commuting varieties via the modality of the space of representations of some given quiver.

We mainly sought to draw the line between dimension vectors whose associated space of representations of the quiver $\widehat{\mathcal{Q}}_s$ has, respectively, zero and positive modality. This turned out to help getting interesting dimension results on the nilpotent commuting variety of parabolics and, eventually, on the nested Hilbert schemes supported at the origin.

One can still look deeper into the above argumentation and more precise results should lead to a better understanding of commuting varieties of parabolics and nested Hilbert schemes. For instance, part 4 of Theorem 6.6, was granted almost for free using our correspondence. Here are a **few promising directions** for further investigation.

- First of all, it would be interesting to get more precise bounds on the modality of the representation $(P, \mathfrak{p}^{nil})$ for any parabolic subalgebra $\mathfrak{p} \subset \mathfrak{gl}_n$. From [GG18], this should allow to make progress on the classification

[72]that is, any orientation of an extended Dynkin diagram in type A,D or E, at dimension vectors corresponding to imaginary roots

[73]and also some other interesting non-maximal

of the commuting varieties $\mathcal{C}(\mathfrak{p})$ into irreducible and reducible cases, as was done in the Borel case. This would also allow to get better estimates for the dimension of the components of $\mathcal{C}(\mathfrak{p})$ and $\mathcal{C}^{nil}(\mathfrak{p})$ and, ultimately, for the corresponding Hilbert schemes.

- Also, progress can be made on the detection of the components of the commuting varieties which still appear at the Hilbert scheme level. That is, the components of $\mathcal{C}^{(nil),cyc}(\mathfrak{p})$. In particular, this would help to make progress on the (ir)reducibility question for $\mathrm{Hilb}^{\boldsymbol{n}}$, especially in the 2-step case $\mathrm{Hilb}^{k,n}$ in which very little is known. For instance, the high dimensional families of commuting pairs constructed for part 4 of Theorem 6.6 do not admit cyclic vectors. The author is not able yet to construct high modality families in $(P_{k,n}, \mathfrak{p}_{k,n}^{(nil)})$ giving rise to components of $\mathcal{C}^{cyc}(\mathfrak{p}_{k,n})$.
- Lastly, it would be interesting to look for similar quiver constructions in types B, C and D. This might require to make use of the notion of *quiver with involution*[74]. One should then also seek whether the associated Hilbert schemes still have a valuable meaning.

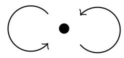
[74]as for other generalisations from type A quiver construction to the other classical cases, see *e.g.* [Sa12]

7 Annexe: other related topics

Up to this point, this dissertation primarily focused on the author's results. The aim of this section is to outline related material in a few directions.

7.1 Doubled setting: a bigger picture

The commuting variety of a Lie algebra $\mathfrak{g}$ is a subvariety of $\mathfrak{g} \times \mathfrak{g}$. This space is equipped with the diagonal G-module structure $g \cdot (x, y) = (g \cdot x, g \cdot y)$. The study of such *doubled setting* is inherently more involved than that of the adjoint action $(G, \mathfrak{g})$.

For instance, when $\mathfrak{g} = \mathfrak{gl}_n$, we get the G-modules associated to the space of represention of the 2-loop quiver in dimension n . This quiver is known to be of wild representation type and its theory of finite dimensional modules in known to be undecidable[75] [Pr88, §17]. In particular, the classification of its representations up to isomorphism, *i.e.* of the GL_n-orbits of $\mathfrak{gl}_n \times \mathfrak{gl}_n$, is hopeless in a certain sense [BS03].

However, all hope is not lost. One way to get an insight into this doubled setting is to describe meaningful sub-G-varieties. In this dissertation, we already discussed the commuting scheme $\mathcal{C}(\mathfrak{g})$ and its nilpotent version $\mathcal{C}^{\mathrm{nil}}(\mathfrak{g})$.

In addition from that, we can mention many other subvarieties of $\mathfrak{g} \times \mathfrak{g}$. The aim of this section is to briefly gather results concerning some selected varieties, thus drawing a bigger picture around the material of this dissertation. Inclusion relations are displayed in Figure 3. For $X \subset \mathfrak{g} \times \mathfrak{g}$, we denote by $X^{nil} := X \cap (\mathfrak{g}^{nil} \times \mathfrak{g}^{nil})$, its intersection with the set of pairs of nilpotent elements.

- The *null-cone*[76] $\mathcal{N}(\mathfrak{g} \times \mathfrak{g})$ is the vanishing locus of the augmentation ideal of $\Bbbk[\mathfrak{g} \times \mathfrak{g}]^{G}$[77]. Its coincides with $G.(\mathfrak{n} \times \mathfrak{n})$ where $\mathfrak{n}$ is the nilradical of a Borel subalgebra of $\mathfrak{g}$, see [KW06]. It should not be confused with the set of pairs of nilpotent elements $\mathfrak{g}^{nil} \times \mathfrak{g}^{nil}$. In particular, $\mathcal{N}(\mathfrak{g} \times \mathfrak{g})$ is irreducible of dimension $3 \dim \mathfrak{n}$.
- The null-cone is an irreducible component of the *nilpotent bicone* $\mathfrak{N}_{\mathfrak{g}}$, which is defined as the scheme whose defining equations are the polarisations of invariant polynomials on $\mathfrak{g}$, see [CM09]. This last scheme is a non-reduced complete intersection. In particular, it is equidimensional of dimension $3 \dim \mathfrak{n}$.

[75] Namely, there cannot exist any algorithm which decides whether each first order statement stated in the language of $\Bbbk[X, Y]$-modules is true for every $\Bbbk[X, Y]$-module

[76] or, more accurately, the *reduced null-cone*

[77] *i.e.* the ideal generated by the homogeneous G-invariant polynomial functions on $\mathfrak{g} \times \mathfrak{g}$ with positive degree

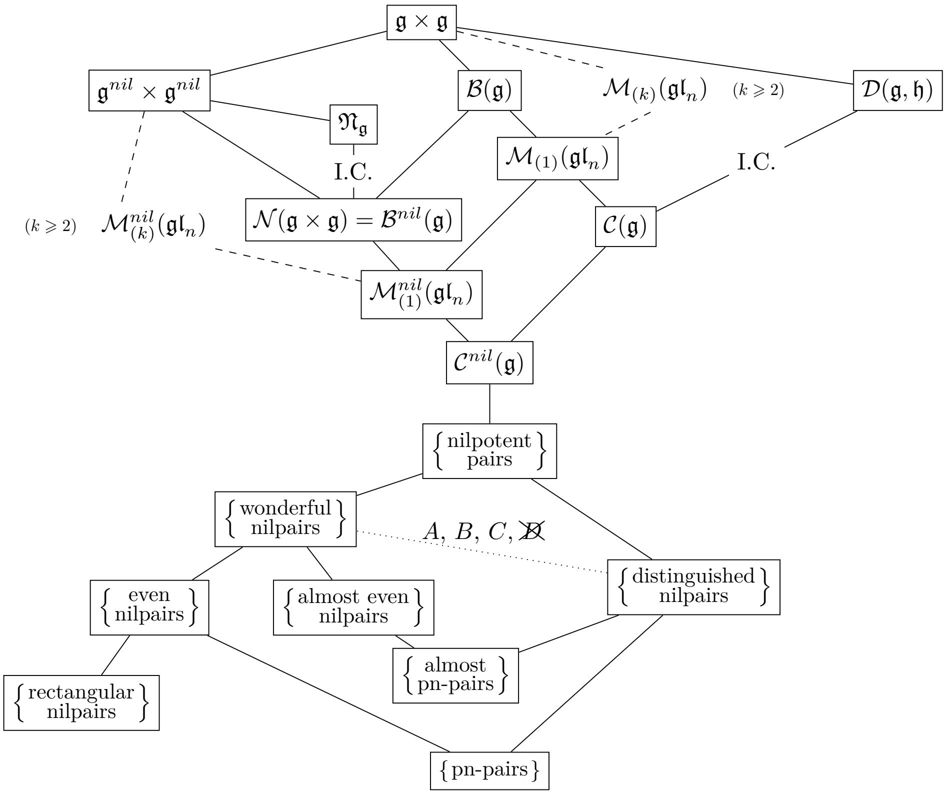


Figure 3: Doubled setting, some G-subsets/subvarieties/subschemes.

Each line denotes an inclusion of the bottom set into the top one, regardless of the scheme structure.

"I.C." means that the bottom variety is an irreducible component of the top one. The dashed lines indicate towers of inclusions $\cdots \subset \mathcal{M}_{(k)} \subset \mathcal{M}_{(k+1)} \subset \ldots$.

The dotted line indicates an inclusion in the classical types A, B, C. This inclusion fails in type D in general

- The *diagonal commutator scheme* $\mathcal{D}(\mathfrak{g}, \mathfrak{h})$ introduced in [Kn05] depends on a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. It is the scheme defined by $\{(x, y) \in \mathfrak{g} \times \mathfrak{g} | \, [x, y] \in \mathfrak{h}\}$. It is a reduced complete intersection and the commuting variety is one of its irreducible components.

- The variety $\mathcal{B}(\mathfrak{g})$ denotes the variety of pairs of elements lying in a common Borel subalgebra[78]. It is irreducible of dimension $3 \dim \mathfrak{n} + 2 \operatorname{rk}(\mathfrak{g})$. We refer to [CZ16] and references therein for further information concerning $\mathcal{B}(\mathfrak{g})$. It is also worth noting that we have an isomorphism of quotients $\mathcal{B}(\mathfrak{g}) /\!\!/ G \cong \mathcal{C}(\mathfrak{g}) /\!\!/ G$.

- When $\mathfrak{g} = \mathfrak{gl}_n$, and $k \in \mathbb{N}$, we define $\mathcal{M}_{(k)}(\mathfrak{gl}_n)$ as the variety of pairs of matrices whose commutator is of rank at most k. When $k = 0$, we get $\mathcal{C}(\mathfrak{g})_{red}$. When $k \geqslant 2$, the variety is irreducible. When $k = 1$, we get an equidimensional variety with $n - 1$ components of dimension $n^2 + 2n - 1$. This last variety is closely linked with the *almost commuting variety* studied in [GG06] $\{(x, y, i, j) \in \mathfrak{gl}_n^2 \times \mathbb{k}^n \times (\mathbb{k}^n)^* | [x, y] + ij = 0\}$. We refer to this article and references therein for all the stated results. The nilpotent version has been studied in [Zo10].

We also pointed out, on the bottom part of the diagram, various sets of pairs of nilpotent commuting elements. The basic notion is that of a *nilpotent pair*, also written *nilpair* in the diagram. By definition, it is a pair of commuting nilpotent elements (e_1, e_2) with $(\mathbb{C}^\times e_1, \mathbb{C}^\times e_2) \subset G \cdot (e_1, e_2)$. Equivalently, (e_1, e_2) admits a *characteristic*, *i.e.* a pair of commuting semisimple elements (h_1, h_2) with $[h_i, e_j] = \delta_{i,j} e_j$, thus mimicking part of the $\mathfrak{sl}_2$-triple machinery[79].

However, even classifying G-orbits of nilpotent pairs turns out to be a wild problem. After the seminal notion of principal nilpotent pair[80], abbreviated as *pn-pair* in the diagram, introduced in [Gi00], the various notions appearing in Diagram 3 have been studied in [Pa01, Pa00, EP01, Yu02]. The set of G-orbits of wonderful nilpotent pairs[81] is in bijection with the set of G-orbits of their associated characteristics and is therefore finite. Thus providing a nice doubled analogue of the notion of a nilpotent element of $\mathfrak{g}$.

We should note that each mentioned set of nilpairs does not form a closed subvariety of $\mathfrak{g} \times \mathfrak{g}$. It would be interesting to look at the closures of these sets and check whether the G-orbits are still classifiable there.

[78] while $\mathcal{C}(\mathfrak{g})$ is the closure of the set of elements lying in a same Cartan subalgebra by Theorem 2.1

[79] The whole $\mathfrak{sl}_2$-triples appear for so-called *rectangular nilpotent pairs*

[80] Namely: a nilpotent pair (e_1, e_2) is principal when it satisfies the regularity condition: "$\dim \mathfrak{g}^{e_1, e_2}$ is of minimal dimension $\dim \mathfrak{h}$", see (3.1).

[81] a characteristic (h_1, h_2) of a nilpotent pair (e_1, e_2) gives rise to a $\mathbb{Q} \times \mathbb{Q}$-grading $\mathfrak{g} = \bigoplus_{(i,j) \in \mathbb{Q} \times \mathbb{Q}} \mathfrak{g}(i, j)$. Then (e_1, e_2) is said to be wonderful when $\dim \mathfrak{g}^{h_1, h_2}$ is equal to $\dim \mathfrak{g}^{e_1, e_2} \cap \bigoplus_{(i,j) \in \mathbb{N} \times \mathbb{N}} \mathfrak{g}(i, j)$.

7.2 Some other directions of generalisation

In the body of this dissertation, we have looked at generalisations from the Lie algebra case to either symmetric Lie algebras, θ-groups or polar representations cases. In Section 7.1, we also introduced several varieties related to commuting varieties. The aim of this section is to provide several other directions of generalisations.

First of all, the doubled setting considered above is a special case of the study of d-tuples. See [Ri88] for early studies in this setting. Concerning commuting varieties, one often either considers the whole scheme $\mathcal{C}^{(d)}(\mathfrak{g})$ of commuting d-tuples, see *e.g.* [NS14], or, in connection with Theorem 2.1, its single principal component $\overline{G \cdot (\mathfrak{h}^d)}$, see *e.g.* [CZ16]. The scheme $\mathcal{C}^{(d)}(\mathfrak{g})$ tends to become reducible when d and/or $\mathrm{rk}(\mathfrak{g})$ increase [NS14]. The Hilbert scheme obtained by considering a quotient associated with $\mathcal{C}^{(d)}(\mathfrak{gl}_n)$ is $Hilb^n(\mathbb{C}^d)$. Concerning reducibility of this scheme, we refer to [HJ18] and references therein.

The presentation made here was mostly valid only over an algebraically closed field with characteristic 0. This setting is one of the most comfortable ones. That is why geometric and representation theories are firstly developed under this assumption. However, the theory is growing faster over other bases, including algebraically closed fields of positive characteristic, *e.g.* [Le02]; the field of real numbers, *e.g.* [Ri79, §6]; finite fields, *e.g.* [BW11]; or rings, *e.g.* [DEG+15].

Multiplicative settings are also of interest. For instance, the commuting variety in the group $GL_n(K)$ can be written as $\{(x, y)|xyx^{-1}y^{-1} = \mathrm{Id}\}$. For reductive groups, the commuting variety is irreducible as shown in [Ri79][82]. Note that the study of sheets in a reductive group is also now quite well understood, see [ACE19] and references therein.

We have already mentioned how to define null-fibers of the moment map and the corresponding symplectic representations in quiver settings. We can qualify this as the *additive* setting. These varieties are much studied in connection with integrable systems which originate from theoretical physics. For instance, the *Calogero-Moser* space can be seen as a *Nakajima quiver variety*[83] corresponding to the one-loop quiver. thus underlining the importance of Theorem 5.5.

Since then, a *multiplicative* setting has been introduced in [CBS06] which is known to be the right setting for other related integrable systems such as Ruijsenaars-Schneider [CF17]. In many important cases, the symplectic reduction is étale-locally isomorphic to one coming from an additive setting [ST18, §7.5]. However, global questions, such as irreducibility, still have to be investigated.

[82] again

[83] here, a variety of the form $\mu^{-1}(\underline{\lambda})/\!\!/G$ for some G-fixed point $\underline{\lambda}$.

With Maxime Fairon (Univ. Glasgow), we are currently working on such questions in the multiplicative case.

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