On commuting varieties and related topics

Habilitation defence

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Plan

$/ \mathbb{C}$

G - variety

$$\mathcal{C}(\mathfrak{g}) := \left\{ (x, y) \in \mathfrak{g} \times \mathfrak{g} \,\middle|\, [x, y] = 0 \right\}$$

$G \overset{ad}{\curvearrowright} \mathfrak{g}$ reductive Lie algebra

e.g.: $\mathfrak{g} = M_n(\mathbb{C})$ $\quad [X, Y] = XY - YX$

$\mathfrak{g} = \text{Lie } G$ $\quad \mathfrak{gl}_n = \text{Lie } GL_n(\mathbb{C})$

alg group

$G = SO(n)$ $\quad \mathfrak{g} = \mathfrak{so}_n = \{\text{antisym mat}\}$

$\mathcal{C}^{nil}(\mathfrak{g}) = \{(x, y) \in \mathcal{C}(\mathfrak{g}) \mid x, y \text{ nilp}\}$

$\mathcal{C}(\mathfrak{p})$

$\mathfrak{p}$ a non-reductive Lie alg

e.g. $\mathfrak{p}$ parabolic subalg of $\mathfrak{g}$

$\mathfrak{g} = \mathfrak{gl}_n \rightsquigarrow \mathfrak{p} = \begin{pmatrix} * & * & * \\ & * & * \\ (0) & & * \\ & & * \end{pmatrix}$

G-variety

$\mathcal{C}(\mathfrak{g}) := \left\{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\right\}$

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"symplectic géo"

alg gp

(H, V) representation

vector space

$\mathcal{C}(V) := \{(x, \varphi) \in V \times V^* \mid \varphi(\mathfrak{h} \cdot x)\}$

μ: moment map

$V \times V^* \to \mathfrak{h}^*$

$\mathcal{C}(V) = \mu^{-1}(0)$

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$: $\mathbb{Z}/2\mathbb{Z}$ grading $\quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$

"symmetric Lie alg"

$G_0 \curvearrowright \mathfrak{g}_1$ $\quad \mathcal{C}(\mathfrak{g}_1)$: G_0-variety

$SO_n \curvearrowright \{\text{symmetric mat}\}$

$\begin{pmatrix} \text{Vinberg} \\ \theta\text{-groups} \\ 2 \rightsquigarrow m \in \mathbb{N}^* \end{pmatrix}$

$\mathcal{C}^{nil}(\mathfrak{p})$

$\mathfrak{p} \subset \mathfrak{gl}_n$

$\mathrm{Hilb}_0^{n_1 \leq - \leq n_k}$

(∗)

[BE16]

[BB19]

$\mathcal{C}^{nil}(\mathfrak{g}) = \{(x,y) \in \mathcal{C}(\mathfrak{g}) \mid x, y \text{ nilp}\}$

$\mathcal{C}^{nil}(\mathfrak{g}_1)$

PhD

G-variety

$\mathcal{C}(\mathfrak{g}) := \{(x,y) \in \mathfrak{g} \times \mathfrak{g} \mid [x,y] = 0\}$

$G \overset{ad}{\curvearrowright} \mathfrak{g}$ reductive Lie algebra

e.g.: $\mathfrak{g} = M_n(\mathbb{C})$ $\quad [x,y] = xy - yx$

$\mathfrak{g} = \mathrm{Lie}\, G$ alg group $\quad$ "$\mathfrak{gl}_n = \mathrm{Lie}\, GL_n(\mathbb{C})$"

$G = SO(n)$ $\quad \mathfrak{g} = \mathfrak{so}_n = \{$antisym mat$\}$

$\mathcal{C}(\mathfrak{p})$

a non-reductive Lie alg

e.g. $\mathfrak{g} \supseteq \mathfrak{p}$ parabolic subalg of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{gl}_n \rightsquigarrow \mathfrak{p} = \begin{pmatrix} * & * & & \\ & * & & * \\ & & * & \\ (0) & & & * \end{pmatrix}$$

(∗)

+ Sheets

[BH16]

+ PhD results

"symplectic géo"

alg gp $\quad$ vector space

(H, V) representation

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Vinberg

θ-groups

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"symmetric Lie alg"

$G_0 \curvearrowright \mathfrak{g}_1$

$SO_n \curvearrowright \{$symmetric mat$\}$

$\mathcal{C}(\mathfrak{g}_1)$: G_0-variety

[Bu11b]

[BLLT17]

[Bu18]

V: Polar

[BB19] M. Boos and M. Bulois, Parabolic conjugation and commuting varieties, *Transformation Groups*, **24** (2019), 951-986.

[Bu11b] M. Bulois, Irregular locus of the commuting variety of reductive symmetric Lie algebras and rigid pairs, *Transformation Groups*, **16** (2011), 1027-1061.

[Bu18] M. Bulois, On the normality of the null-fiber of the moment map for θ- and tori representations, *Journal of Algebra*, **507** (2018), 502-524.

[BE16] M. Bulois and L. Evain, Nested punctual Hilbert schemes and commuting varieties of parabolic subalgebras, *Journal of Lie Theory*, **26** (2016), 497-533.

[BH16] M. Bulois and P. Hivert, Sheets in symmetric Lie algebras and slice induction, *Transformation Groups*, **21** (2016), 355-375.

[BLLT17] M. Bulois, C. Lehn, M. Lehn and R. Terpereau, Towards a symplectic version of the Chevalley restriction theorem, *Compositio Mathematica*, **153** (2017), 647- 666.

Questions :

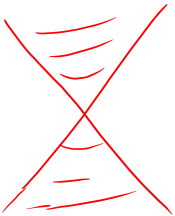
∗ structure (dim, irreducible comp)

∗ geometry (smoothness, normality ...)

∗ invariant theory (quotients)

Structure of $\mathcal{C}(\mathfrak{g})$

singular



normal cone ?

$\dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \mathrm{rk}\, \mathfrak{g}$

e.g.: $\mathfrak{g} = \mathfrak{sl}_2 \rightsquigarrow \dim \mathcal{C}(\mathfrak{g}) = 4$

$\dim \mathfrak{g} \times \mathfrak{g} = 6$

Structure of $\mathcal{C}(\mathfrak{g})$

vector space

$(x, y) \in \mathcal{C}(\mathfrak{g}) \Leftrightarrow y \in \mathfrak{g}^x = \{ z \mid [x ; z] = 0 \} \rightsquigarrow (x, \mathfrak{g}^x)$

$\underbrace{G.(x, \mathfrak{g}^x)}_{\dim \mathfrak{g}} \subset \mathcal{C}(\mathfrak{g})$

if $\#\{G\text{-orbits}\} < +\infty \rightsquigarrow \mathcal{C}(\mathfrak{g}) = \bigsqcup_{G.x} \underbrace{G(x, \mathfrak{g}^x)}_{\dim \mathfrak{g}}$

$(\mathfrak{z}(\mathfrak{g}^x)^{reg}, \mathfrak{g}^x) \subset \mathcal{C}(\mathfrak{g})$

$z \in$

$\dim \mathfrak{g}^z = \dim \mathfrak{g}^x \Longleftrightarrow \dim G.x = \dim G_z$

$\dim(\mathfrak{z}(\mathfrak{g}^x)^{reg} /\!/ G) + \dim \mathfrak{g}$

$G.(\mathfrak{z}(\mathfrak{g}^x)^{reg}, \mathfrak{g}^x) \subset \mathcal{C}(\mathfrak{g})$

$J = G(\mathfrak{z}(\mathfrak{g}^x)^{reg})$

$\mathcal{C}(\mathfrak{g}) = \bigsqcup_J \overline{G.(\mathfrak{z}(\mathfrak{g}^x)^{reg}, \mathfrak{g}^x)}$

Plan

1. Commuting variety
2. Sheets
3. Hilbert schemes

Definition

Let (H, V) be a representation and $m \in \mathbb{N}$. Let

$$V^{(m)} := \{v \in V \mid \dim H.v = m\}$$

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The **sheets** of V are the irreducible components of the $V^{(m)}$

$\mathcal{C}_S(\mathfrak{g}) := \{(x, y) \in \mathcal{C}(\mathfrak{g}) \mid x \in S\}$

S sheet irred of dim $\dim \mathfrak{g} + \dim (S/\!/G)$

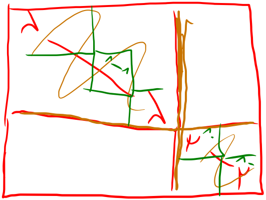
$= \dim \mathfrak{g} + (\dim S - m)$

Sheets of a Lie algebras

Definition

The decomposition class of $x = s + n$ is Jordan decomposition

$$J(x) := G.\{\mathfrak{z}(\mathfrak{g}^s)^{reg} + n\} =: J(\mathfrak{l}, \mathcal{O})$$



$\mathcal{O}$

$\mathfrak{l}$ Levi   nilp orbit in $\mathfrak{l}$

λ, μ varying

$\dim J(x) /\!/ G = \dim \mathfrak{z}(\mathfrak{g}^s)$

Sheets of a Lie algebras

Definition

The decomposition class of $x = s + n$ is

$$J(x) := G.\{\mathfrak{z}(\mathfrak{g}^s)^{reg} + n\} =: J(\mathfrak{l}, \mathcal{O})$$

Theorem (Borho, Kraft, Im-Hof, Petersen)

1. $\#\{\text{Dec. classes}\} < \infty$ and dec. classes are locally closed varieties, irreducibles and smooth.

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2. S sheet $\Rightarrow S = \bigsqcup_{\text{dec. cl.}} J$ and $\overline{S} = \bigsqcup_{\text{dec cl.}} J$. $\exists J$ s.t. $\overline{J} = \overline{S}$
3. $J(\mathfrak{l}_1, \mathcal{O}_1) \subset \overline{J(\mathfrak{l}_2, \mathcal{O}_2)}^{reg} \Leftrightarrow Ind_{\mathfrak{l}_2}^{\mathfrak{l}_1}(\mathcal{O}_2) = \mathcal{O}_1$. $\mathfrak{l}_2 \subseteq \mathfrak{l}_1$ up to G-conjugacy

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4.
   - Smoothness in classical types.
   - 1 singular example for G_2.

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$J(\mathfrak{l}, \mathcal{O})$ dense in S

Corollary

Sheets are classified by pairs $(\mathfrak{l}, \mathcal{O})$ with $\mathcal{O}$ rigid in $\mathfrak{l}$. $\rightsquigarrow \forall\, \mathfrak{l}_2 \subset \mathfrak{l}$ $\forall\, \mathcal{O}_2$ L_2-orbit $Ind_{\mathfrak{l}_2}^{\mathfrak{l}}(\mathcal{O}_2) \neq \mathcal{O}$

Sheets of a symmetric Lie algebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$G_0 \circlearrowleft \mathfrak{g}_1$

Definition

The decomposition class of $x = s + n \in \mathfrak{g}_1$ is

$\mathfrak{l}$ Levi in $\mathfrak{g}$

$$J_{G_0}(x) := G_0.\{\mathfrak{z}_{\mathfrak{g}_1}(\mathfrak{g}^s)^{reg} + n\} =: J_{G_0}(\mathfrak{l}, \mathcal{O})$$

L_0-orbit in $\mathfrak{l}_1$

Theorem (Tauvel, Yu, Lebarbier, Bulois (PhD))

1. $\#\{\text{Dec. classes}\} < \infty$ and dec. classes are locally closed varieties, irreducibles and smooth.
2. S sheet $\Rightarrow S = \bigsqcup_{\text{dec. cl.}} J$ and $\overline{S} = \bigsqcup_{\text{dec cl.}} J$. (over $\mathbb{C}$)
3. $J(\mathfrak{l}_1, \mathcal{O}_1) \subset \overline{J(\mathfrak{l}_2, \mathcal{O}_2)}^{reg} \Leftrightarrow Ind_{\mathfrak{l}_2}^{\mathfrak{l}_1}(\mathcal{O}_2) = \mathcal{O}_1$.
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   - 1 singular example for G_2.

Corollary

Sheets are classified by some pairs $(\mathfrak{l}, \mathcal{O})$ with $\mathcal{O}$ rigid in $\mathfrak{l}$.

Sheets of a symmetric Lie algebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

Definition

The decomposition class of $x = s + n \in \mathfrak{g}_1$ is

$$J_{G_0}(x) := G_0.\{\mathfrak{z}_{\mathfrak{g}_1}(\mathfrak{g}^s)^{reg} + n\} =: J_{G_0}(\mathfrak{l}, \mathcal{O})$$

Theorem (Bulois-Hivert [BH16])

1. $\#\{\text{Dec. classes}\} < \infty$ and dec. classes are locally closed varieties, irreducibles and smooth.
2. S sheet $\Rightarrow S = \bigsqcup_{\text{dec. cl.}} J$ and $\overline{S} = \bigsqcup_{\text{dec cl.}} J$. (over $\mathbb{K}$)
3. $J_{G_0}(\mathfrak{l}_1, \mathcal{O}_1) \subset \overline{J_{G_0}(\mathfrak{l}_2, \mathcal{O}_2)}^{reg} \Leftrightarrow Ind_{\mathfrak{l}_2}^{\mathfrak{l}_1}(\mathcal{O}_2) = \mathcal{O}_1$.
4.
   - Smoothness in classical types.
   - 1 singular example for G_2.

Corollary

Sheets are classified by pairs $(\mathfrak{l}, \mathcal{O})$ with $\mathcal{O}$ rigid in $\mathfrak{l}$.

$\mathfrak{l}_1 = \mathfrak{g}$

$$\underbrace{G_0.e}_{\mathcal{O}} = J_{G_0}(\mathfrak{g}, G_0.e) \subset \overline{J_{G_0}(\mathfrak{l}_2, \mathcal{O}_2)}^{reg} \quad \Leftrightarrow \quad Ind_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathcal{O}_2) = G_0.e.$$

Following Kostant, Slodowy, Katsylo, Im-Hof

$$G_0.e = J_{G_0}(\mathfrak{g}, G_0.e) \subset \overline{J_{G_0}(\mathfrak{l}_2, \mathcal{O}_2)}^{reg} \Leftrightarrow Ind_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathcal{O}_2) = G_0.e.$$

$$\Leftrightarrow (e + U) \cap J_{G_0}(\mathfrak{l}_2, \mathcal{O}_2) \neq \emptyset$$

for some affine U slice $\subset V$

$U = \mathfrak{g}^f$ (e, h, f) sl_2-triple

Following Kostant, Slodowy, Katsylo, Im-Hof

$$G_0.e = J_{G_0}(\mathfrak{g}, G_0.e) \subset \overline{J_{G_0}(\mathfrak{l}_2, \mathcal{O}_2)}^{reg} \Leftrightarrow Ind_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathcal{O}_2) = G_0.e.$$

$$\Leftrightarrow (e+U) \cap J_{G_0}(\mathfrak{l}_2, \mathcal{O}_2) \neq \emptyset$$

for some affine U

Note : $(G_0 . e) \cap (e+U) = e$

Work in Progress:

$\rightarrow$ Algorithm to find rigid orbits $\rightarrow$ equations for $\mathfrak{g}^{(\leq m)}$ $\rightsquigarrow$ equation $\mathfrak{g}^{(\leq m)} \cap e+U$

$\rightsquigarrow$ if vanishing locus $\{e\} \Rightarrow G_0.e$ rigid

$\rightarrow$ Algorithm for induction? $\rightarrow$ same with eq of $\overline{J(\mathfrak{l}_2, \mathcal{O}_2)}$

? $\rightarrow e \in$

$\rightarrow$ Geometry of sheets

S sheet $\quad G \times (\underbrace{(e+U)\cap S}) \rightarrow S$ smooth map

$\rightarrow$ some singular sheets in each exceptional case.

Plan

1. Commuting variety
2. Sheets
3. **Hilbert schemes**

Definition

The Hilbert scheme (of n points in $\mathbb{A}^2$) is the scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ represented by the functor

plane

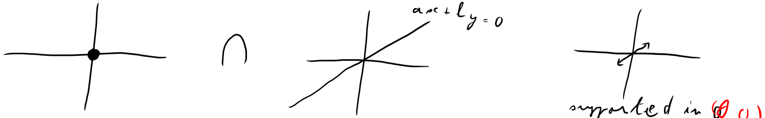
$$\boldsymbol{H}^n(S) := \left\{ Z \subset \mathbb{A}^2 \times S \;\middle|\; \begin{pmatrix} Z \\ \downarrow \\ S \end{pmatrix} \text{ flat, surjective of degree } n \right\}$$

Case $S = Spec(\mathbb{C})$

$Z = Spec(\mathbb{C}[x,y]/I)$ ← dim n

$\mathrm{Hilb}^n(\mathbb{A}^2) = \{ Z \text{ 0-dim subscheme of } \mathbb{A}^2 \mid \mathrm{length}(Z) = n \}$

e.g. $Z_{a,b} = Spec(\mathbb{C}[x,y]/(x^2, y^2, xy, ax + by))$



$\mathrm{Hilb}^2_0(\mathbb{A}^2) = \mathbb{P}^1$

0 → supported at $(0,0)$

Hilb	Hilb^n	Hilb_0^n	$\mathrm{Hilb}^{n_1,\leqslant\cdots\leqslant n_k}$	$\mathrm{Hilb}_0^{n_1\leqslant\cdots\leqslant n_k}$
$\mathcal{C}$	$\mathcal{C}(\mathfrak{gl}_n)$	$\mathcal{C}^{nil}(\mathfrak{gl}_n)$	$\mathcal{C}(\mathfrak{p})$	$\mathcal{C}^{nil}(\mathfrak{p})$
G	GL_n	GL_n	P	P
reference	*Nakajima* 1999	*Baranovski* 2001	*Bulois – Evain* 2016	

"nested" $\{ \underbrace{Z_1}_{\text{length } n_1} \subset \cdots \subset \underbrace{Z_k}_{n_k} \subset \mathbb{A}^2 \}$

$Z \subset \mathbb{A}^2$ $\quad Z = \mathrm{Spec}(\underbrace{\mathbb{K}[x,y]/I}_{\dim n})$ $\quad \cdot x \; X$, $\cdot y \; Y$

$\rightsquigarrow \underbrace{X, Y}_{\in \mathcal{C}(\mathfrak{gl}_n)} \in \mathrm{End}(\mathbb{K}[x,y]/I)$

Hilb	Hilb^n	Hilb^n_0	$\mathrm{Hilb}^{n_1,\leqslant\cdots\leqslant n_k}$	$\mathrm{Hilb}_0^{n_1\leqslant\cdots\leqslant n_k}$
$\mathcal{C}$	$\mathcal{C}(\mathfrak{gl}_n)$	$\mathcal{C}^{nil}(\mathfrak{gl}_n)$	$\mathcal{C}(\mathfrak{p})$	$\mathcal{C}^{nil}(\mathfrak{p})$
G	GL_n	GL_n	P	P
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Proposition

Let $\mathcal{C}$, Hilb and G be as in Table. $\mathcal{C}(\mathfrak{g})$

$\tilde{\mathcal{C}}^{cyc}(\mathfrak{gl}_n) = \{((\overbrace{x, y}^{\psi}), \underset{\mathbb{C}^n}{v}) \mid v \text{ cyclic for } (x, y)\}$

$\mathcal{C}^{cyc}(\mathfrak{gl}_n) = \{(x, y) \mid \exists\, v \text{ cyclic for } (x, y)\}$

Then

- Hilb $\cong \tilde{\mathcal{C}}^{cyc}/G$.
- Set theoretically Hilb $\cong \mathcal{C}^{cyc}/G$, inducing a bijection between
  - $*$ $\{$ irr. comp. of Hilb of dim. m $\}$
  - $*$ $\{$ irr. comp. of $\mathcal{C}^{cyc}$ of dim. $m + (\dim G - n)\}$. $\subset \{$ irr comp of $\mathcal{C}$ $\}$

Chain for the study of $Hilb_0^{n_1\leqslant\cdots\leqslant n_k}$

$$\begin{tikzcd}
 & Hilb_0^{n_1\leq\cdots\leq n_k} \arrow[dl, leftrightarrow, "1:1"'] \\
\mathcal{C}^{nil}(\mathfrak{p}) \arrow[d, leftrightarrow, "1:1"] & \\
\text{sheets of } \mathfrak{p}^{nil} &
\end{tikzcd}$$

If $\#\{P\text{-orbits in } \mathfrak{p}^{nil}\} \Longrightarrow \dim \mathcal{C}^{nil} = \dim \mathfrak{p} - 1$

Theorem (BB19)

Classification of all parabolics with $\#\{\text{P-orbits in } \mathfrak{p}^{nil}\} < +\infty$.

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Classification of all parabolics with $\#\{\text{P-orbits in } \mathfrak{p}^{nil}\} < +\infty$.

1. In these cases, $\dim \mathcal{C}^{nil}(\mathfrak{p}) = \dim \mathfrak{p} - 1$ and $\dim \mathrm{Hilb}_0^{n_1 \leqslant \cdots \leqslant n_k} = n_k - 1$.

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   $n_k - n_1 \leq n_k - n_2 \leq \cdots \leq n_k - n_{k-1}$

2. If $\mathfrak{p}$ is maximal parabolic, $6 \leqslant n_1 \leqslant n_2 - 6$, then $\dim \mathcal{C}^{nil}(\mathfrak{p}) \geqslant \dim \mathfrak{p}$ and $\dim \mathrm{Hilb}_0^{n_1, n_2} \geqslant n_2$.

$\mathfrak{p} = \begin{pmatrix} * & * \\ & * \end{pmatrix}$

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Classification of all parabolics with $\#\{\text{P-orbits in } \mathfrak{p}^{nil}\} < +\infty$.

1. In these cases, $\dim \mathcal{C}^{nil}(\mathfrak{p}) = \dim \mathfrak{p} - 1$ and $\dim \mathrm{Hilb}_0^{n_1 \leqslant \cdots \leqslant n_k} = n_k - 1$.
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3. $\mathcal{C}(\mathfrak{p})$ might be reducible, even for $\mathfrak{p}$ maximal parabolic.

Proposition

There is a bijection

$$\left\{ P\text{-orbits in } \mathfrak{p}^{nil} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \prod_i GL_{n_i}\text{-orbits of representations of } \mathcal{Q}_s \\ \text{with dimension vector } (n_1, \ldots, n_k), \\ \text{satisfying the relations} \\ \beta_{i+1}\alpha_i = \alpha_i\beta_i, \text{each } \beta_i \text{ is nilpotent}, \\ \text{and each } \alpha_i \text{ is injective.} \end{array} \right\}$$

$$\mathcal{Q}_s: \quad \begin{tikzcd} \underset{1}{\bullet} \arrow[loop above, "\beta_1"] \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \arrow[loop above, "\beta_2"] \arrow[r, "\alpha_2"] & \underset{3}{\bullet} \arrow[loop above, "\beta_3"] \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{s-3}"] & \underset{s-2}{\bullet} \arrow[loop above, "\beta_{s-2}"] \arrow[r, "\alpha_{s-2}"] & \underset{s-1}{\bullet} \arrow[loop above, "\beta_{s-1}"] \arrow[r, "\alpha_{s-1}"] & \underset{s}{\bullet} \arrow[loop above, "\beta_s"] \end{tikzcd}$$

β_s = nilp or $\in \mathfrak{gl}_{n_k}$

n_1 n_2 n_k

Thank you.