Real structures on nilpotent orbit closures

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Abstract.   We determine the equivariant real structures on nilpotent orbits and the normalizations of their closures for the adjoint action of a complex semisimple algebraic group on its Lie algebra.
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1.   Introduction

Let G be a complex semisimple algebraic group, and let $\mathfrak{g}$ be its Lie algebra. Let G_0 be a *real form* of G, that is, a real algebraic group such that $G_0 \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \simeq G$ as complex algebraic groups. Given a complex quasi-projective G-variety X, we say that a real G_0-variety X_0 is an $(\mathbb{R}, G_0)$-*form* of X if $X_0 \times_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) \simeq X$ as complex G-varieties. Let us note that certain G-varieties admit no $(\mathbb{R}, G_0)$-form. On the other hand, when an $(\mathbb{R}, G_0)$-form exists, there may be several (and even uncountably many) which are pairwise non-isomorphic.

The goal of this article is to address the following basic question.

Question 1. *What are the $(\mathbb{R}, G_0)$-forms of the nilpotent orbit closures in $\mathfrak{g}$ and of their normalizations?*

To give a real form G_0 of G is equivalent to giving a *real group structure* σ on G, that is, an antiregular involution on G compatible with the group structure, and giving an $(\mathbb{R}, G_0)$-form of X is equivalent to giving a (G, σ)-*equivariant real structure* μ on X, that is, an antiregular involution on X such that

$$\forall g \in G, \ \forall x \in X, \ \ \mu(g \cdot x) = \sigma(g) \cdot \mu(x). \qquad (\star)$$

Moreover, two $(\mathbb{R}, G_0)$-forms are isomorphic if and only if the two corresponding (G, σ)-equivariant real structures are *equivalent*, i.e. conjugate by a G-equivariant automorphism of X. (See Sections 2.1 and 2.2 for more details on these notions.)

With this terminology, Question 1 can be rephrased as follows.

Question 2. *What are the (G, σ)-equivariant real structures on the nilpotent orbit closures in $\mathfrak{g}$ and on their normalizations?*

If X is an *almost homogeneous* G-variety, that is, if X contains a dense open G-orbit U, then the condition $(\star)$ implies that $\mu(U) = U$, and thus $\mu_{|U}$

is a (G, σ)-equivariant real structure on U. Therefore, a natural strategy to determine the equivariant real structures on nilpotent orbit closures and their normalizations is in two steps. First, we determine the equivariant real structures on nilpotent orbits. Second, we determine which ones extend to their closures and their normalizations.

Denoting by $d\sigma_e \colon \mathfrak{g} \to \mathfrak{g}$ the differential of $\sigma \colon G \to G$ at the identity element, we check (see Example 2.8) that $d\sigma_e$ is a (G, σ)-equivariant real structure on $\mathfrak{g}$, viewed as a G-variety for the adjoint action. In particular, if $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$, then $d\sigma_e$ induces a (G, σ)-equivariant real structure on $\mathcal{O}$ if and only if $d\sigma_e(\mathcal{O}) = \mathcal{O}$. However, there are also equivariant real structures on nilpotent orbits that are not obtained by differentiating a real group structure on G (Example 1.1), nor even by restricting an equivariant real structure from the Lie algebra $\mathfrak{g}$ (Example 1.2).

Example 1.1.   Let $\mathfrak{g}$ be a semisimple Lie algebra, and assume that $d\sigma_e(\mathcal{O}) = \mathcal{O}$. Let $\theta \in \mathbb{R}$. Then $\mu_\theta \colon \mathcal{O} \to \mathcal{O}$, $v \mapsto e^{i\theta} d\sigma_e(v)$ is a (G, σ)-equivariant real structure on $\mathcal{O}$ which is not obtained by differentiating a real group structure on G when $\theta \notin 2\pi\mathbb{Z}$ (because, in this case, μ_θ does not preserve the Lie bracket).

Example 1.2.   Let $G = \mathrm{SL}_3(\mathbb{C})$ with $\sigma(g) = \overline{g}$ for all $g \in G$ (here $\overline{g}$ denotes the complex conjugate of g), and let $\mathcal{O}_{reg}$ be the *regular* nilpotent orbit in $\mathfrak{sl}_3$ (i.e. the unique nilpotent orbit whose closure contains all the other nilpotent orbits). Then the map μ defined by

$$\mathcal{O}_{reg} \to \mathcal{O}_{reg}, \ g \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \left(\sigma(g) \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \sigma(g) \cdot \begin{bmatrix} 0 & 1 & i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is a (G, σ)-equivariant real structure on $\mathcal{O}_{reg}$ that does not extend to a (G, σ)-equivariant real structure on $\mathfrak{sl}_3$ (see Section 6 for a proof of this claim).

The following theorem provides a complete answer to the second part of Question 2. More precisely, we determine the equivariant structures on the nilpotent orbits and the normalizations of their closures. In the cases where the closure is itself not normal, it remains an open question to determine all the equivariant real structures. This will be discussed in Section 5.

Main Theorem. *Let G be a complex semisimple algebraic group endowed with a real group structure σ. Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$, let $\overline{\mathcal{O}}$ be the closure of $\mathcal{O}$ in $\mathfrak{g}$, and let $\widetilde{\mathcal{O}}$ be the normalization of $\overline{\mathcal{O}}$. Then the following are equivalent:*

(a) $\mathcal{O}$ is $d\sigma_e$-stable (i.e. $d\sigma_e(\mathcal{O}) = \mathcal{O}$);
(b) $\mathcal{O}$ admits a (G, σ)-equivariant real structure;
(c) $\widetilde{\mathcal{O}}$ admits a (G, σ)-equivariant real structure; and
(d) $\sigma_D(w(\mathcal{O})) = w(\mathcal{O})$,

where

- *σ_D is the Dynkin diagram automorphism of $\mathrm{Dyn}(G)$ induced by σ (see Definition 2.5); and*
- *$w(\mathcal{O})$ is the weighted Dynkin diagram associated to $\mathcal{O}$ (see [10, Section 3.5]).*

Moreover, if these equivalent assertions hold, then all (G, σ)-equivariant real structures on $\mathcal{O}$, resp. on $\widetilde{\mathcal{O}}$, are equivalent.

The Main Theorem will be proved in several steps as follows:

- $(a) \Leftrightarrow (b)$ is Proposition 3.1.
- $(b) \Leftrightarrow (c)$ is the first part of Proposition 5.2 (see also Section 2.2.2).
- $(a) \Leftrightarrow (d)$ is Corollary 3.3.
- If the equivalent conditions (a)-(d) of the Main Theorem hold, then the fact that all (G, σ)-equivariant real structures on $\mathcal{O}$ are equivalent is Theorem 4.2, and the fact that all (G, σ)-equivariant real structures on $\widetilde{\mathcal{O}}$ are equivalent is the second part of Proposition 5.2.

Let us make a few comments related to Question 2 and our Main Theorem:

(1) In Section 3, we provide the list of nilpotent orbits that admit a (G, σ)-equivariant real structure. It turns out that, except for a few cases in type D_{2n} (with $n \geq 2$), every nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ admits a (G, σ)-equivariant real structure when $\mathfrak{g}$ is simple.

(2) For every nilpotent orbit $\mathcal{O}$ in a complex simple Lie algebra $\mathfrak{g}$, we have $\mathcal{O}^{d\sigma_e} = \mathcal{O} \cap \mathfrak{g}^{d\sigma_e}$, which is a real manifold (possibly empty) whose $G_0(\mathbb{R})$-orbits, usually called *real nilpotent orbits*, are classified (see [10, Section 9]). Let us mention that a study of some properties of the complexification process for these real nilpotent orbits can be found in [12].

(3) When $\overline{\mathcal{O}}$ is non-normal, we do not know whether every (G, σ)-equivariant real structure on $\mathcal{O}$ extends to $\overline{\mathcal{O}}$; see Example 5.3. In particular, we did not manage to provide a complete answer to the first part of Question 2. A brief review on what is known about the (non-)normality of nilpotent orbit closures in semisimple Lie algebras can be found in Section 5.

(4) There are examples of almost homogeneous varieties for which equivalent real structures on the open orbit extend to inequivalent real structures on the whole variety; see e.g. the case of $\mathrm{SL}_2(\mathbb{C})$-threefolds studied in [23, Section 3].

Notation. In this article we work over the field of real numbers $\mathbb{R}$ and over the field of complex numbers $\mathbb{C}$. We denote by

$$\Gamma := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$$

the Galois group of the field extension $\mathbb{C}/\mathbb{R}$.

We will always denote by G a complex algebraic group, by $\mathrm{Dyn}(G)$ its Dynkin diagram (when G is semisimple), and by G_0 a real algebraic group such that $G_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \simeq G$. When X is a complex algebraic variety, the group of automorphisms of X over $\mathrm{Spec}(\mathbb{C})$ is denoted by $\mathrm{Aut}_{\mathbb{C}}(X)$. Moreover, when X is equipped with a regular G-action, the subgroup of G-equivariant automorphisms is denoted by $\mathrm{Aut}_{\mathbb{C}}^{G}(X)$. We denote by $\mathbb{G}_{m,\mathbb{C}}$ the multiplicative group over $\mathbb{C}$.

The reader is referred to [10] for the general background on Lie theory and nilpotent orbits in semisimple Lie algebras.

2. Preliminaries

2.1. Real group structures.

2.1.1.   We start by recalling the notions of real forms and real group structures for a given complex algebraic group G, and how these two notions are related.

Definition 2.1.

(i) A *real form* of G is a pair (G_0, Θ) with G_0 a real algebraic group and $\Theta \colon G \to G_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ an isomorphism of complex algebraic groups. (Most of the time, one drops the isomorphism Θ and simply write that G_0 is a form of G.)

(ii) A *real group structure* σ on G is a scheme involution on G such that the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\;\;\sigma\;\;} & G \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \bar{z})} & \mathrm{Spec}(\mathbb{C})
\end{array}
$$

commutes and

$$
\iota_G \circ \sigma = \sigma \circ \iota_G \quad \text{and} \quad m_G \circ (\sigma \times \sigma) = \sigma \circ m_G,
$$

where $\iota_G \colon G \to G$ is the inverse morphism and $m_G \colon G \times G \to G$ is the multiplication morphism.

(iii) Two real group structures σ_1 and σ_2 on G are *equivalent* if there exists a complex algebraic group automorphism $\psi \in \mathrm{Aut}_{\mathrm{gr}}(G)$ such that

$$
\sigma_2 = \psi \circ \sigma_1 \circ \psi^{-1}.
$$

There is a correspondence between real group structures on G and real forms of G given as follows (see [1, Section 2.12] for a general statement of Galois descent, and [13, Section 1.4] for the particular case of algebraic groups).

- To a real group structure σ on G, one associates the real algebraic group $G_0 := G/\langle \sigma \rangle$, where γ acts on G through σ, and the isomorphism Θ is given by (q, f), where $q \colon G \to G_0$ is the quotient morphism and $f \colon G \to \mathrm{Spec}(\mathbb{C})$ is the structure morphism.

- To a real form (G_0, Θ) of G, one associates the real group structure σ that makes the following diagram commutes:

$$
\begin{array}{ccc}
G & \xrightarrow{\;\Theta\;} & G_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \\
\sigma \downarrow & & \downarrow {\scriptstyle Id \times (\lambda \mapsto \bar{\lambda})} \\
G & \xrightarrow{\;\Theta\;} & G_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})
\end{array}
$$

Moreover, two real forms of G are isomorphic (as real algebraic groups) if and only if the corresponding real group structures are equivalent.

2.1.2.   In this article, we are mostly interested in complex semisimple algebraic groups; nevertheless, the case of complex tori will play a role in the proof of Theorem 4.2.

Lemma 2.2.   *(Real group structures on complex tori; see [24, Lemma 1.5] and [9, Theorem 2].) Let $T \simeq \mathbb{G}_{m,\mathbb{C}}^n$ be an n-dimensional complex torus with $n \geq 1$.*
  (i) *If $n = 1$, then T has exactly two inequivalent real group structures, defined by $\sigma_0 : t \mapsto \bar{t}$ and $\sigma_1 : t \mapsto \bar{t}^{-1}$.*
  (ii) *If $n = 2$, then $\sigma_2 : (t_1, t_2) \mapsto (\overline{t_2}, \overline{t_1})$ defines a real group structure on T.*
  (iii) *If $n \geq 2$, then every real group structure on T is equivalent to exactly one real group structure of the form $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$, where $n = n_0 + n_1 + 2n_2$.*

2.1.3.   Let G be a complex semisimple algebraic group. There exists a central isogeny $\widetilde{G} \to G$, where $\widetilde{G}$ is a simply-connected semisimple algebraic group. Then $\widetilde{G}$ is isomorphic to a product of simply-connected simple algebraic groups (see [11, Exercise 1.6.13 and Section 6.4]). Moreover, every real group structure σ on G lifts uniquely to a real group structure $\widetilde{\sigma}$ on $\widetilde{G}$ (this follows for instance from [25, Section 3.4, Theorem]).

The next result reduces the classification of real group structures on simply-connected semisimple algebraic groups to the classification of real group structures on simply-connected simple algebraic groups.

Lemma 2.3.   *([24, Lemma 1.7]) Let σ be a real group structure on a complex simply-connected semisimple algebraic group $\widetilde{G} \simeq \prod_{i \in I} G_i$, where the G_i are the simple factors of $\widetilde{G}$. Then, for a given $i \in I$, we have the following possibilities:*
  (i) *$\sigma(G_i) = G_i$ and $\sigma_{|G_i}$ is a real group structure on G_i; or*
  (ii) *there exists $j \neq i$ such that $\sigma(G_i) = G_j$, then $G_i \simeq G_j$ and $\sigma_{|G_i \times G_j}$ is equivalent to $(g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$ for an arbitrary real group structure σ_0 on $G_i \simeq G_j$.*

Real group structures on complex simply-connected simple algebraic groups are well-known (see e.g. [14, Section 1.7.2] or [17, Section VI.10]). Therefore, all real group structures on complex (simply-connected) semisimple algebraic groups can be determined from Lemma 2.3.

2.1.4.   Let G be a complex semisimple algebraic group endowed with a real group structure σ. In order to describe which nilpotent orbits in $\mathfrak{g}$ admit a (G, σ)-equivariant real structure (in Section 3), we need to recall the notions of split real group structures, as well as the notions of inner and outer twists.
Definition 2.4.
  (i) If there exists a Borel subgroup $B \subseteq G$ such that $\sigma(B) = B$, then σ is called *quasi-split*. If furthermore B contains a maximal torus T such that $\sigma(T) = T$ and the restriction $\sigma_{|T}$ is equal to the product $\sigma_0^{\times \dim(T)}$ (with the notation of Lemma 2.2), then σ is called *split*.

(ii) For $c \in G$, we write
$$\mathrm{inn}_c \colon G \to G, \; g \mapsto cgc^{-1}.$$

We say that inn_c is an *inner* automorphism of G. An automorphism of G that is not inner is called *outer*. If σ_1 and σ_2 are two real group structures on G, we may write $\sigma_2 = \varphi \circ \sigma_1$ with $\varphi \in \mathrm{Aut}_{\mathrm{gr}}(G)$, and we say that σ_2 is an *inner* (resp. an *outer*) *twist* of σ_1 if φ is an inner (resp. an outer) automorphism of G. Note that the relation of being inner twists is an equivalence relation on the set of real group structures on G.

According to [11, Proposition 7.2.12], every real group structure on G is an inner twist of a unique (up to equivalence) quasi-split real group structure on G. Moreover, in types $A_1 - B - C - E_7 - E_8 - F_4 - G_2$ (i.e. when $\mathrm{Aut}(\mathrm{Dyn}(G))$ is trivial), any quasi-split real group structure is equivalent to a split one, and in types $A_n(n \geq 2) - D - E_6$ (i.e. when $\mathrm{Aut}(\mathrm{Dyn}(G))$ contains exactly one conjugacy class of elements of order 2), there are exactly two equivalence classes of quasi-split real group structures; this follows for instance from [11, Propositions 7.2.2 and 7.2.12].

2.1.5.   Finally, we recall how the choice of a real group structure σ on a complex semisimple algebraic group G induces a Γ-action on $\mathrm{Dyn}(G)$. We refer to [11, Remark 7.1.2] or [24, Section A.2] and references therein for details.

Let $B \subseteq G$ be a Borel subgroup, and let $T \subseteq B$ be a maximal torus. (We say that (T, B) is a *Borel pair* in G.) Then $(T', B') := (\sigma(T), \sigma(B))$ is another Borel pair in G, and so there exists $c_\sigma \in G$ (unique up to left multiplication by an element of T) such that

$$c_\sigma T' c_\sigma^{-1} = T \quad \text{and} \quad c_\sigma B' c_\sigma^{-1} = B.$$

Let $\theta := \mathrm{inn}_{c_\sigma} \circ \sigma$ that is an antiregular automorphism of G. Since $\theta(T) = T$ and $\theta(B) = B$, the automorphism θ induces a lattice automorphism of the character group $\mathbb{X}(T) := \mathrm{Hom}_{\mathrm{gr}}(T, \mathbb{G}_{m,\mathbb{C}})$ as follows:

$$\forall \chi \in \mathbb{X}(T), \; \chi \mapsto {}^\gamma\chi := \sigma_0 \circ \chi \circ \theta^{-1},$$

where $\sigma_0(t) = \bar{t}$ is the complex conjugation on $\mathbb{G}_{m,\mathbb{C}}$. One can check that this lattice automorphism is of order ≤ 2, that it does not depend on the choice of c_σ, and that it stabilizes the sets of roots $\mathcal{R} \subseteq \mathbb{X}(T)$ and of simple roots $\mathcal{S} \subseteq \mathcal{R}$ associated with the triple (G, B, T).

Definition 2.5. By the previous discussion, the choice of σ induces an action $\Gamma \to \mathrm{Aut}(\mathrm{Dyn}(G))$, usually referred as the $\star$-*action induced by* σ. This action does not depend on the choice of the Borel pair (T, B) in G. We will denote by $\sigma_D \in \mathrm{Aut}(\mathrm{Dyn}(G))$ the image of the non-trivial element $\gamma \in \Gamma$.

Remark 2.6.   It follows from the definition of the $\star$-action that if σ is an inner twist of σ', then the $\star$-actions induced by σ and σ' coincide. In particular, if σ is an inner twist of a split real group structure on G, then σ_D is trivial. On the other hand, if σ is an outer twist of a split real group structure on G, then it follows from [11, Section 1.5, second half of p. 41] that σ_D is non-trivial.

2.2. Equivariant real structures.   We fix a complex algebraic group G, a real group structure σ on G, and we denote by $G_0 = G/\langle\sigma\rangle$ the corresponding real form.

2.2.1.   We start by recalling the notions of real forms and equivariant real structures for complex G-varieties, and how they are related, as we did for complex algebraic groups in Section 2.1.1.

Definition 2.7. Let X be a complex G-variety.

(i) An $(\mathbb{R}, G_0)$-*form* of X is a pair (X_0, Ξ) with X_0 a real G_0-variety and $\Xi\colon X \to X_0 \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ an isomorphism of complex G-varieties. (As for real forms of complex algebraic groups, one usually drops the isomorphism Ξ and simply write that X_0 is a form of X.)

(ii) A (G, σ)-*equivariant real structure* on X is an antiregular involution μ on X, that is, a scheme involution on X such that the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ \ \ \ \ \mu\ \ \ \ \ \ } & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathbb{C}) & \xrightarrow{\ \mathrm{Spec}(z\mapsto\overline{z})\ } & \mathrm{Spec}(\mathbb{C})
\end{array}
$$

and satisfying the condition $(\star)$ stated in the introduction.

(iii) Two (G, σ)-equivariant real structures μ and μ' on X are *equivalent* if there exists a G-equivariant automorphism $\varphi \in \mathrm{Aut}^G_{\mathbb{C}}(X)$ such that $\mu' = \varphi \circ \mu \circ \varphi^{-1}$.

We now suppose that X is quasi-projective. (Later in this article, we will in fact restrict to the case where X is quasi-affine.) Under this condition, as for real group structures (see Section 2.1.1), there is a correspondence between (G, σ)-equivariant real structures on X and $(\mathbb{R}, G_0)$-forms of X (see [3, Section 5]). Moreover, this correspondence induces a bijection between isomorphism classes of $(\mathbb{R}, G_0)$-forms of X (as real algebraic G_0-varieties) and equivalence classes of (G, σ)-equivariant real structures on X.

Example 2.8.   Considering the Lie algebra $\mathfrak{g}$ of G as a G-variety for the adjoint action, we observe that the antilinear Lie algebra involution $d\sigma_e$, obtained by differentiating σ at the identity element, is also a (G, σ)-equivariant real structure on $\mathfrak{g}$. Indeed, for a fixed $g \in G$, if we denote $\mathrm{inn}_g\colon G \to G, h \mapsto ghg^{-1}$, then we have

$$\mathrm{inn}_{\sigma(g)} \circ \sigma = \sigma \circ \mathrm{inn}_g,$$

and the condition $(\star)$ follows from computing the differential at the identity element on both sides of this equality.

2.2.2.   The condition $(\star)$ implies that an equivariant real structure μ on a G-variety X maps G-orbits to G-orbits, and so a given G-orbit is either fixed by μ or exchanged with another G-orbit. In particular, if X has a dense open orbit U, then U must be fixed by μ, and so $\mu_{|U}$ is an equivariant real structure on U. Let us note that a given equivariant real structure μ_0 on U need not extend to X, but if μ_0 extends to X, then this extension is unique by [15, Chapter I, Lemma 4.1].

2.2.3.   In the case of homogeneous spaces, we have the following criterion for the existence of an equivariant real structure.

Lemma 2.9.   ([24, Lemma 2.4]) *Let $X = G/H$ be a homogeneous space. Then X admits a (G, σ)-equivariant real structure if and only if there exists $g_0 \in G$ such that the following two conditions hold:*
(i) (G, σ)-*compatibility condition:* $\sigma(H) = g_0 H g_0^{-1}$
(ii) *involution condition:* $\sigma(g_0) g_0 \in H$
in which case one real structure is given by

$$\forall k \in G, \ \mu(kH) = \sigma(k) g_0 H.$$

2.2.4.   The classical way to determine the number of equivalence classes for real structures, assuming that such a structure exists, is via Galois cohomology (see [26] for a general reference on Galois cohomology).

Lemma 2.10.   ([24, Lemma 2.11], see also [33, Section 8])
Let X be a G-variety endowed with a (G, σ)-equivariant real structure μ_0. Let the Galois group Γ act on $\mathrm{Aut}_\mathbb{C}^G(X)$ by conjugation by μ_0, and denote by $\mathrm{H}^1(\Gamma, \mathrm{Aut}_\mathbb{C}^G(X))$ the corresponding first Galois cohomology set. Then the map

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, \mathrm{Aut}_\mathbb{C}^G(X)) & \to & \{equivalence\ classes\ of\ (G, \sigma)\text{-}equivariant\ real\ structures\ on\ X\} \\ \varphi & \mapsto & \varphi \circ \mu_0 \end{array}$$

is a bijection that sends the identity element to the equivalence class of μ_0.

2.3. Nilpotent orbits in semisimple Lie algebras.

In this section we recall some classical facts concerning nilpotent orbits in semisimple Lie algebras that we will need to prove the Main Theorem.

Let G be a complex semisimple algebraic group, and let $\mathfrak{g}$ be its Lie algebra. We denote by $\mathcal{N}$ the *nilpotent cone* of $\mathfrak{g}$, that is, the G-stable closed subset of $\mathfrak{g}$ formed by the elements $v \in \mathfrak{g}$ such that $\mathrm{ad}_v \colon \mathfrak{g} \to \mathfrak{g}, u \mapsto [u, v]$, is a nilpotent endomorphism of the $\mathbb{C}$-vector space $\mathfrak{g}$. Then the following hold:

(N1) The nilpotent cone $\mathcal{N}$ is a finite union of G-orbits (see [10, Corollary 3.2.15]).

(N2) The dimension of a nilpotent orbit is always even (see [10, Corollary 1.4.8]).

(N3) Each nilpotent orbit is stable for the $\mathbb{G}_{m,\mathbb{C}}$-action induced by the scalar multiplication on $\mathfrak{g}$ (see the proof in [10, Lemma 4.3.1]).

(N4) To every nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}$ is associated a unique weighted Dynkin diagram $w(\mathcal{O})$, that is, a labeling of the nodes of the Dynkin diagram $\mathrm{Dyn}(G)$ with labels in $\{0, 1, 2\}$ (see [10, Section 3.5]). When $\mathfrak{g}$ is a complex simple Lie algebra, a complete classification of the weighted Dynkin diagrams corresponding to nilpotent orbits can be found in [10, Section 5.3], for the classical cases, and in [10, Section 8.4], for the exceptional cases.

(N5) If $v \in \mathcal{N}$, then $(G \cdot v) \cap \mathfrak{z}(\mathfrak{g}^v) \subseteq \mathfrak{g}$ is a dense open subset of $\mathfrak{z}(\mathfrak{g}^v)$, where $\mathfrak{g}^v := \{u \in \mathfrak{g} \mid [u, v] = 0\}$ and $\mathfrak{z}(\mathfrak{g}^v)$ is the center of $\mathfrak{g}^v$ (see [30, Propositions 35.3.3 and 35.3.4]).

(N6) Let $v_1, v_2 \in \mathcal{N}$. Then $\mathrm{Stab}_G(v_1)$ and $\mathrm{Stab}_G(v_2)$ are conjugate in G if and only if v_1 and v_2 belong to the same nilpotent orbit (see [5, Theorem 3.7.1], using the fact that each nilpotent orbit is a decomposition class).

(N7) Since the Lie algebra $\mathfrak{g}$ is semisimple, it is isomorphic to a direct sum of simple Lie algebras $\bigoplus_{i\in I}\mathfrak{g}_i$ on which G acts componentwise through the adjoint group $G/Z(G)$. Then the nilpotent orbits in $\mathfrak{g}$ identify with the products $\prod_{i\in I}\mathcal{O}_i$, where each $\mathcal{O}_i$ is a nilpotent orbit in $\mathfrak{g}_i$. The same holds for the nilpotent orbit closures in $\mathfrak{g}$ and their normalizations.

3. Existence of real structures on nilpotent orbits

Let G be a complex semisimple algebraic group endowed with a real group structure σ. Let (T, B) be a Borel pair in G. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$.

In this section we prove the equivalences $(a) \Leftrightarrow (b)$ (Proposition 3.1) and $(a) \Leftrightarrow (d)$ (Corollary 3.3) of the Main Theorem. In particular, the equivalence $(b) \Leftrightarrow (d)$ gives a combinatorial criterion for the existence of an equivariant real structure on a given nilpotent orbit, and we then use this criterion to determine which nilpotent orbits admit an equivariant real structure (see Proposition 3.5 and the discussion before).

Proposition 3.1.
(i) *If $d\sigma_e(\mathcal{O}) = \mathcal{O}$, then $(d\sigma_e)_{|\mathcal{O}}$ is a (G,σ)-equivariant real structure on $\mathcal{O}$.*
(ii) *If $d\sigma_e(\mathcal{O}) \neq \mathcal{O}$, then $\mathcal{O}$ does not admit a (G,σ)-equivariant real structure.*

Proof. (i): Follows from Example 2.8.
(ii): Let $v \in \mathcal{O}$ and $v' := d\sigma_e(v) \in \mathcal{O}' := d\sigma_e(\mathcal{O})$. Let $H := \mathrm{Stab}_G(v)$ and $H' := \mathrm{Stab}_G(v')$. Since $d\sigma_e$ is a (G,σ)-equivariant real structure on $\mathfrak{g}$, the condition $(\star)$ holds for every element in $\mathfrak{g}$; in particular, the condition $(\star)$ applied to v implies that $\sigma(H) = H'$. Let us assume that $\mathcal{O}$ admits a (G,σ)-equivariant real structure. By Lemma 2.9(i), there exists $g_0 \in G$ such that $\sigma(H) = g_0 H g_0^{-1}$, and so H and $H' = \sigma(H)$ are conjugate. But, according to (N6), the subgroups H and H' are conjugate if and only if $\mathcal{O} = \mathcal{O}'$. The result follows by contraposition. $\blacksquare$

Proposition 3.2. *Let $w(\mathcal{O})$ be the weighted Dynkin diagram associated to $\mathcal{O}$. Then*

$$w(d\sigma_e(\mathcal{O})) = \sigma_D(w(\mathcal{O})),$$

where σ_D is the Dynkin diagram automorphism induced by σ (see Definition 2.5).

Proof. We keep the notation of Section 2.1.5. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be the Cartan subalgebra associated to $T \subseteq G$. Given a nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}$, we recall the construction of the corresponding weighted Dynkin diagram $w(\mathcal{O})$ (see [10, Section 3.5] for details). There exists an $\mathfrak{sl}_2$-triple (x, t, y) with $x \in \mathcal{O}$, $t \in \mathfrak{t}$, and such that for every simple root $\alpha \in \mathcal{S}$, we have $d\alpha_e(t) \in \mathbb{N}$, where $d\alpha_e \in \mathfrak{t}^*$ is the differential of the simple root α at the identity element. In fact, for a given $\mathcal{O}$, there is a unique $t \in \mathfrak{t}$ satisfying this last condition, and one can furthermore show that $d\alpha_e(t) \in \{0, 1, 2\}$. Then $w(\mathcal{O})$ is defined by associating to each node $\alpha \in \mathcal{S}$ of $\mathrm{Dyn}(G)$ the label $d\alpha_e(t) \in \{0, 1, 2\}$.

Let us now consider the weighted Dynkin diagram $\sigma_D(w(\mathcal{O}))$. By definition

of σ_D, it is the weighted Dynkin diagram defined by associating to each node $\alpha \in \mathcal{S}$ of $\mathrm{Dyn}(G)$ the label

$$d(\,^\gamma\alpha)_e(t) = d\alpha_e(d\theta_e^{-1}(t)) \in \{0, 1, 2\}, \tag{A}$$

where $d\theta_e^{-1}$ is the antilinear Lie algebra automorphism obtained by differentiating $\theta^{-1} = \sigma \circ \mathrm{inn}_{c_\sigma^{-1}} : G \to G$ at the identity element. On the other hand, if (x, t, y) is an $\mathfrak{sl}_2$-triple as above, then

$$(x', t', y') := (d\theta_e^{-1}(x), d\theta_e^{-1}(t), d\theta_e^{-1}(y))$$

is an $\mathfrak{sl}_2$-triple with $x' \in d\theta_e^{-1}(\mathcal{O}) = d\sigma_e(c_\sigma^{-1} \cdot \mathcal{O}) = d\sigma_e(\mathcal{O})$, $t' \in \mathfrak{t}$ (since $\theta(T) = T$), and such that for every simple root $\alpha \in \mathcal{S}$, we have $d\alpha_e(t') \in \mathbb{N}$ by (A). It follows that $w(d\sigma_e(\mathcal{O})) = \sigma_D(w(\mathcal{O}))$. ∎

The following result is a straightforward consequence of Proposition 3.2 together with (N4).

Corollary 3.3. *We have $d\sigma_e(\mathcal{O}) = \mathcal{O}$ if and only if $\sigma_D(w(\mathcal{O})) = w(\mathcal{O})$.*

We now determine which nilpotent orbits in $\mathfrak{g}$ admit an equivariant real structure (i.e. the nilpotent orbits $\mathcal{O}$ such that $d\sigma_e(\mathcal{O}) = \mathcal{O}$). Applying Lemma 2.3, we reduce to the case where either

(A) $\mathfrak{g}$ is simple; or

(B) $\mathfrak{g} \simeq \mathfrak{l} \oplus \mathfrak{l}$ with $\mathfrak{l} = \mathrm{Lie}(L)$ a simple summand, and $\sigma : L \times L \to L \times L$ is $(g_1, g_2) \mapsto (\sigma'(g_2), \sigma'(g_1))$, where σ' is any real group structure on L.

In Case (B), the nilpotent orbits of $\mathfrak{g}$ stabilized by $d\sigma_e$ are the orbits of the form $(\mathcal{O}_1, d\sigma'_e(\mathcal{O}_1))$ with $\mathcal{O}_1$ a nilpotent orbit in $\mathfrak{l}$. Hence, only these nilpotent orbits admit a (G, σ)-equivariant real structure.

On the other hand, Case (A) will be handled by Proposition 3.5, but we first need to recall the partition type classification of nilpotent orbits in type D_n.

Theorem 3.4. ([10, Theorem 5.1.4]) *Nilpotent orbits in $\mathfrak{so}_{2n}$ (type D_n with $n \geq 4$) are parametrized by partitions of $2n$ in which even parts occur with even multiplicity, except that "very even" partitions $\mathbf{d} = [d_1, \ldots, d_r]$ (those with only even parts, each having even multiplicity) correspond to two orbits, denoted by $\mathcal{O}_{\mathbf{d}}^I$ and $\mathcal{O}_{\mathbf{d}}^{II}$.*

Proposition 3.5. *Assume that G is simple. Then $d\sigma_e(\mathcal{O}) \neq \mathcal{O}$ if and only if one of the following two conditions holds:*

(i) *G is of type D_4, σ is an outer twist of a split real group structure on G (i.e. $\mathfrak{g}^{d\sigma_e} \simeq \mathfrak{so}_{5,3}$ or $\mathfrak{so}_{7,1}$), and $\mathcal{O}$ belongs to one of the pairs of orbits swapped by the $\star$-action induced by σ (see Example 3.6 below).*

(ii) *G is of type D_{2n} with $n \geq 3$, σ is an outer twist of a split real group structure on G (i.e. $\mathfrak{g}^{d\sigma_e} \simeq \mathfrak{so}_{p,q}$ with p and q odd), and $\mathcal{O} \in \{\mathcal{O}_{\mathbf{d}}^I, \mathcal{O}_{\mathbf{d}}^{II}\}$ with $\mathbf{d}$ a very even partition.*

Proof.

- If σ is an inner twist of a split real group structure on G, then the corresponding Dynkin diagram automorphism σ_D is trivial (see Remark 2.6). Hence, Corollary 3.3 yields $d\sigma_e(\mathcal{O}) = \mathcal{O}$ in this case.
- In types A_n $(n \geq 2)$, D_{2n+1} $(n \geq 2)$ and E_6, the weighted Dynkin diagram $w(\mathcal{O})$ is invariant under the non-trivial Dynkin diagram automorphism; see [10, Lemma 3.6.5] for the type A_n, [10, Lemma 5.3.4 and Remark 5.3.6] for the type D_{2n+1}, and [10, Section 8.4] for the case E_6. Hence, Corollary 3.3 yields again $d\sigma_e(\mathcal{O}) = \mathcal{O}$ in this case.
- In type D_{2n} $(n \geq 2)$, if σ is an outer twist of a split real group structure on G, then σ_D is a non-trivial involution of $\mathrm{Dyn}(G)$ (see Remark 2.6). In type D_4, the group $\mathrm{Aut}(\mathrm{Dyn}(G)) \simeq \mathfrak{S}_3$ contains three (conjugate) involutions, and σ_D can be each of them depending on the choice of σ in its equivalence class. In type D_{2n} $(n \geq 3)$, we have $\mathrm{Aut}(\mathrm{Dyn}(G)) = \langle \sigma_D \rangle \simeq \mathbb{Z}/2\mathbb{Z}$. The result follows then from the description of the weighted Dynkin diagrams given in [10, Example 5.3.7] in type D_4 and in [10, Lemmas 5.3.4 and 5.3.5, Remark 5.3.6] in type D_{2n} $(n \geq 3)$.

■

Example 3.6. A partial order relation is defined on the set of nilpotent orbits of $\mathfrak{g}$ by inclusion of closures. In type D_4, the Hasse diagram representing this partial order is the following (see [10, Section 6.2]).
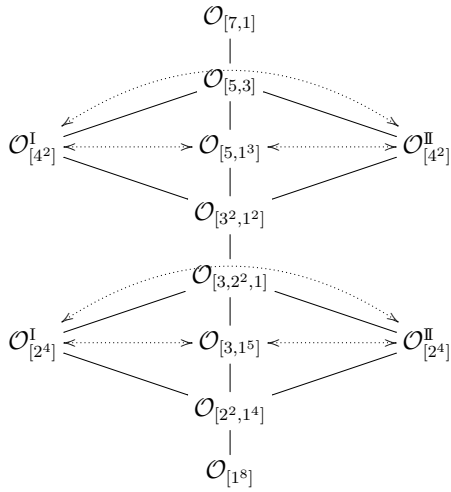
Here the dotted arrows indicate which pairs of orbits can be swapped by the $\star$-action induced by an outer twist of a split real group structure on G.

4. Uniqueness of real structures on nilpotent orbits

We keep the notation of Section 3. In this section, we prove that all (G, σ)-equivariant real structures on $\mathcal{O}$ are equivalent (Theorem 4.2).

We recall that, if $X \simeq G/H$ is a homogeneous space, then there is an isomorphism of groups (see [31, Proposition 1.8]) given by

$$N_G(H)/H \xrightarrow{\sim} \mathrm{Aut}_{\mathbb{C}}^G(X), \; nH \mapsto (\varphi_{nH} \colon gH \mapsto gn^{-1}H),$$

where $N_G(H)$ denotes the normalizer of H in G.

Let $\mathcal{O} \simeq \prod_{i \in I} \mathcal{O}_i \simeq G/H$ be a nilpotent orbit in $\mathfrak{g} \simeq \bigoplus_{i \in I} \mathfrak{g}_i$, and let

$Q := N_G(H)/H \simeq \mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O})$. The following result, which is essentially due to Brylinski and Kostant [8], describes the structure of the algebraic group Q.

Theorem 4.1. *The linear algebraic group Q is connected and solvable. Furthermore, the unipotent radical U of Q is of codimension*

$$d = d(\mathcal{O}) := \mathrm{Card}(\{i \in I, \ \mathcal{O}_i \neq \{0\}\})$$

in Q. In particular, we have $\mathbb{T} := Q/U \simeq \mathbb{G}_{m,\mathbb{C}}^d$ and $Q \simeq U \rtimes \mathbb{T}$.

Proof. Let $v \in \mathcal{O}$ such that $\mathrm{Stab}_G(v) = H$. Then $Q \simeq Q \cdot v$ as algebraic varieties. Also, by [8, Theorem D], we have $Q \cdot v = \mathcal{O} \cap \mathfrak{z}(\mathfrak{g}^v)$, which is a dense open subset of $\mathfrak{z}(\mathfrak{g}^v)$ by (N5). Hence, $Q \simeq Q \cdot v$ is irreducible, and so Q is connected.

Now the fact that Q is solvable and that its unipotent radical is of codimension d in Q is precisely the content of [8, Theorem 23 and Remark 23.1]. The last sentence follows from general results on the structure of connected solvable linear algebraic groups (see e.g. [2, Section 10.6]). ∎

Theorem 4.2. *All (G, σ)-equivariant real structures on $\mathcal{O}$ are equivalent.*

Proof. If $\mathcal{O}$ admits no (G, σ)-equivariant real structure, then there is nothing to prove. Otherwise, according to Proposition 3.1, $\mu_0 := (d\sigma_e)_{|\mathcal{O}}$ is a (G, σ)-equivariant real structure on $\mathcal{O}$. Thus, conjugation by μ_0 yields a Γ-action on the group $\mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O}) \simeq Q$, and according to Lemma 2.10, the first Galois cohomology set $\mathrm{H}^1(\Gamma, Q)$ parametrizes the equivalence classes of (G, σ)-equivariant real structures on $\mathcal{O}$. Therefore proving the theorem is equivalent to prove that this set is a singleton.

Let $K := \mathbb{G}_{m,\mathbb{C}}^d$ acting on $\mathfrak{g} \simeq \bigoplus_{i \in I} \mathfrak{g}_i$ by scalar multiplication on each simple component $\mathfrak{g}_i$ such that $\mathcal{O}_i \neq \{0\}$. By (N3), the torus K acts faithfully on $\mathcal{O}$, and so there is an inclusion of algebraic groups $K \hookrightarrow \mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O}) \simeq Q$. Using the fact that $\mu_0 = d\sigma_e$ is an antilinear Lie algebra involution that permutes the $\mathfrak{g}_i$, we have the following two possibilities (with the notation of Lemma 2.2):

- Assume that $\mathcal{O}_i \neq \{0\}$ and $\mu_0(\mathfrak{g}_i) = \mathfrak{g}_i$. In this case, Γ acts on the corresponding $\mathbb{G}_{m,\mathbb{C}}$ through σ_0. Indeed, denoting $\varphi_t \colon \mathfrak{g}_i \to \mathfrak{g}_i, \ v \mapsto t \cdot v$, we see that

$$\mu_{0|\mathfrak{g}_i} \circ \varphi_t \circ \mu_{0|\mathfrak{g}_i} = \varphi_{\bar{t}}.$$

- Assume that $\mathcal{O}_i, \mathcal{O}_j \neq \{0\}$ and $\mu_0(\mathfrak{g}_i \oplus \mathfrak{g}_j) = \mathfrak{g}_j \oplus \mathfrak{g}_i$. Then $\mathfrak{g}_i \oplus \mathfrak{g}_j \simeq \mathfrak{l} \oplus \mathfrak{l}$, for some simple Lie algebra $\mathfrak{l}$, and $\mu_{0|\mathfrak{g}_i \oplus \mathfrak{g}_j}$ identifies with $\mathfrak{l} \oplus \mathfrak{l} \to \mathfrak{l} \oplus \mathfrak{l}, \ (v_1, v_2) \mapsto (\psi(v_2), \psi^{-1}(v_1))$, where ψ is an antilinear Lie algebra automorphism. In this case, Γ acts on the corresponding $\mathbb{G}_{m,\mathbb{C}}^2$ through the real group structure σ_2. Indeed, denoting $\varphi_{(t_1,t_2)} \colon \mathfrak{l} \oplus \mathfrak{l} \to \mathfrak{l} \oplus \mathfrak{l}, \ (v_1, v_2) \mapsto (t_1 \cdot v_1, t_2 \cdot v_2)$, we see that

$$\mu_{0|\mathfrak{g}_i \oplus \mathfrak{g}_j} \circ \varphi_{(t_1,t_2)} \circ \mu_{0|\mathfrak{g}_i \oplus \mathfrak{g}_j} = \varphi_{(\overline{t_2}, \overline{t_1})}.$$

Therefore, the Γ-action on $\mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O})$, given by the conjugation by μ_0, stabilizes the subgroup K, and the restriction of the Γ-action on $\mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O})$ to $K = \mathbb{G}_{m,\mathbb{C}}^d$

coincides with the Γ-action on K induced by a real group structure equivalent to $\sigma_0^{\times n_0} \times \sigma_2^{\times n_2}$ for some $n_0, n_2 \in \mathbb{Z}_{\geq 0}$ such that $d = n_0 + 2n_2$. But then [24, Proposition 1.18] implies that $\mathrm{H}^1(\Gamma, K)$ is trivial.

Moreover, the homomorphism $Q \to Q$, $q \mapsto \gamma \cdot q$ is an antiregular algebraic group involution, and so the Γ-action on Q stabilizes the unipotent radical U of Q and induces a Γ-action on the quotient $\mathbb{T} = Q/U$ such that the quotient morphism $\pi \colon Q \to \mathbb{T}$ is then Γ-equivariant. It follows that isomorphism of algebraic groups $\pi_{|K} \colon K \simeq \mathbb{T}$ is Γ-equivariant. Consequently, it induces an isomorphism of abelian groups between $\mathrm{H}^1(\Gamma, K)$ and $\mathrm{H}^1(\Gamma, \mathbb{T})$, and so $\mathrm{H}^1(\Gamma, \mathbb{T})$ is trivial.

Finally, the exact sequence of algebraic groups given by Theorem 4.1, namely

$$1 \to U \to Q \to \mathbb{T} \to 1,$$

induces a long exact sequence of pointed sets (see [26, Chp. I, Section 5.5]). In particular, we have

$$\mathrm{H}^1(\Gamma, U) \to \mathrm{H}^1(\Gamma, Q) \to \mathrm{H}^1(\Gamma, \mathbb{T}) = \{e\}.$$

But $\mathrm{H}^1(\Gamma, U)$ is a singleton by [26, Chp. III, Section 2.1, Proposition 6], and so $\mathrm{H}^1(\Gamma, Q)$ is also a singleton, which concludes the proof of the theorem. ∎

5.   Extension of real structures to nilpotent orbit closures

We maintain the notation of Section 3. In this section, we prove that every (G, σ)-equivariant real structure on $\mathcal{O}$ extends uniquely to $\widetilde{\mathcal{O}}$, and that all (G, σ)-equivariant real structures on $\widetilde{\mathcal{O}}$ are equivalent (Proposition 5.2). We then briefly review what is known about the (non-)normality of nilpotent orbit closures in semisimple Lie algebras, and finish this section by considering an example related to our extension problem (Example 5.3).

Lemma 5.1.   *Let X be a complex normal affine G-variety, and assume that there exists a G-stable dense open subset $X_0 \subseteq X$ such that $\mathrm{codim}(X \backslash X_0, X) \geq 2$. Then any G-equivariant automorphism of X_0 extends uniquely to a G-equivariant automorphism of X.*

Proof.   It is clear that if an automorphism extends from X_0 to X, then this extension is unique by [15, Chapter I, Lemma 4.1].

Since X is normal and $\mathrm{codim}(X \setminus X_0, X) \geq 2$, the map

$$H^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X_0, \mathcal{O}_{X_0}), \ f \mapsto f_{|X_0}$$

is an isomorphism of $\mathbb{C}$-algebras, and so $\mathrm{Spec}(\mathrm{H}^0(X_0, \mathcal{O}_{X_0})) \simeq \mathrm{Spec}(\mathrm{H}^0(X, \mathcal{O}_X)) \simeq X$. By [29, Tag 01P9], the affinization morphism

$$X_0 \to \mathrm{Spec}(\mathrm{H}^0(X_0, \mathcal{O}_{X_0})) \simeq X$$

is an open immersion; it coincides with the inclusion morphism $X_0 \hookrightarrow X$.

Let φ be an automorphism of X_0. Then it follows that

$$\tilde{\varphi} := \mathrm{Spec}(\varphi^*) \colon X \simeq \mathrm{Spec}(\mathrm{H}^0(X_0, \mathcal{O}_{X_0})) \to \mathrm{Spec}(\mathrm{H}^0(X_0, \mathcal{O}_{X_0})) \simeq X$$

is an automorphism of X that extends (uniquely) φ. Furthermore, the set

$$\bigcap_{g \in G} \{x \in X \mid \tilde{\varphi}(g \cdot x) = g \cdot \tilde{\varphi}(x)\}$$

is a closed subset of X containing the dense open subset X_0, and so it is equal to X. Therefore, $\tilde{\varphi}$ is a G-equivariant automorphism of X. ∎

Proposition 5.2. *Every (G, σ)-equivariant real structure on $\mathcal{O}$ extends uniquely to the normalization $\widetilde{\mathcal{O}}$ of $\overline{\mathcal{O}}$. Furthermore, all (G, σ)-equivariant real structures on $\widetilde{\mathcal{O}}$ are equivalent.*

Proof. If μ is a (G, σ)-equivariant real structure on $\mathcal{O}$, then by Theorem 4.2, there exists $\varphi \in \mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O})$ such that $\mu = \varphi \circ (d\sigma_e)_{|\mathcal{O}} \circ \varphi^{-1}$. The equivariant real structure $(d\sigma_e)_{|\mathcal{O}}$ is the restriction of $d\sigma_e \colon \mathfrak{g} \to \mathfrak{g}$, so it extends (uniquely by [15, Chapter I, Lemma 4.1]) to $\overline{\mathcal{O}}$, and also to $\widetilde{\mathcal{O}}$ by the universal property of the normalization morphism (see e.g. [14, Section 12.11]). With a slight abuse of notation, we will write $(d\sigma_e)_{|\widetilde{\mathcal{O}}}$ to denote the extension of $(d\sigma_e)_{|\mathcal{O}}$ to $\widetilde{\mathcal{O}}$.

In addition, $\varphi \in \mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O})$ also extends (uniquely) to $\tilde{\varphi} \in \mathrm{Aut}_{\mathbb{C}}^G(\widetilde{\mathcal{O}})$ by Lemma 5.1 applied to $X = \widetilde{\mathcal{O}}$ and $X_0 = \mathcal{O}$ (here the fact that $\mathrm{codim}(X \setminus X_0, X) \geq 2$ follows from (N2)). Hence, $\tilde{\mu} := \tilde{\varphi} \circ (d\sigma_e)_{|\widetilde{\mathcal{O}}} \circ \tilde{\varphi}^{-1}$ is a (G, σ)-equivariant real structure on $\widetilde{\mathcal{O}}$ extending μ. The fact that this extension is uniquely defined by μ follows again from [15, Chapter I, Lemma 4.1].

It remains to prove the second part of the proposition. Let $\widetilde{\mu}$ be any (G, σ)-equivariant real structure on $\widetilde{\mathcal{O}}$. Then its restriction to $\mathcal{O}$, denoted by μ, is equivalent to $(d\sigma_e)_{|\mathcal{O}}$ by Theorem 4.2. Hence, there exists $\varphi \in \mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O})$ such that $\mu = \varphi \circ (d\sigma_e)_{|\mathcal{O}} \circ \varphi^{-1}$. By Lemma 5.1, φ extends to $\widetilde{\varphi} \in \mathrm{Aut}_{\mathbb{C}}^G(\widetilde{\mathcal{O}})$, and so we must have $\widetilde{\mu} = \widetilde{\varphi} \circ (d\sigma_e)_{|\widetilde{\mathcal{O}}} \circ \widetilde{\varphi}^{-1}$. In particular, $\widetilde{\mu}$ is equivalent to $(d\sigma_e)_{|\widetilde{\mathcal{O}}}$. Therefore, all (G, σ)-equivariant real structures on $\widetilde{\mathcal{O}}$ are equivalent. ∎

The question of which nilpotent orbits in a complex simple Lie algebra have normal closure has been studied by many authors. Let us note that, since a product variety is normal if and only if each factor is normal, the (non-)normality of $\overline{\mathcal{O}} \simeq \prod_{i \in I} \overline{\mathcal{O}}_i$ (see (N7)) is therefore easily deduced from the (non-)normality of the $\overline{\mathcal{O}}_i$, which means that we can reduce to the case where $\mathfrak{g}$ is simple. Here is a list of some known results on this topic, with references:

- The regular orbit, the subregular orbit, and the minimal orbit have normal closure [18, 32, 7].
- Nilpotent orbits in $\mathfrak{sl}_n$ have normal closure [16, 20].
- Nilpotent orbits in $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ having normal closure are determined in [21, 28].
- Nilpotent orbits in $\mathfrak{g}_2$, $\mathfrak{f}_4$, and $\mathfrak{e}_6$ having normal closure are classified in [22, 19, 6, 27]. A list of nilpotent orbits in $\mathfrak{e}_7$ and $\mathfrak{e}_8$ whose closure is non-normal, which is expected to be the complete list, is given in [4, Section 7.8].

We now give an example of a quasi-affine surface X_0 in $\mathbb{A}_{\mathbb{C}}^4$, endowed with

a real structure μ that does not extend to the closure X of X_0 in $\mathbb{A}^4_{\mathbb{C}}$ but does extend to the normalization $\widetilde{X}$ of X.

Example 5.3. Let X be the image of the morphism

$$f\colon \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^4_{\mathbb{C}},\ (s,t) \mapsto (s, st, t^2, t^3);$$

it is a closed non-normal affine surface in $\mathbb{A}^4_{\mathbb{C}}$, defined by the prime ideal

$$(u^2w - v^2, vw - ux, uw^2 - vx, w^3 - x^2) \text{ in } \mathbb{C}[u,v,w,x].$$

One can check that X has an isolated singularity at $p := (0,0,0,0)$, and that f restricts to an isomorphism $\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\} \to X_0 := X \setminus p$. Moreover, the normalization of X, which coincides with the affinization of X_0, is $\widetilde{X} = \mathbb{A}^2_{\mathbb{C}}$, and the normalization morphism corresponds to the inclusion of $\mathbb{C}$-algebras $\mathbb{C}[y, yz, z^2, z^3] \subseteq \mathbb{C}[y, z]$.

Let $\tilde{\mu}$ be the real structure on $\widetilde{X}$ given by $(s,t) \mapsto (\bar{t}, \bar{s})$. It restricts to a real structure μ on X_0 (as $(0,0)$ is fixed by $\tilde{\mu}$). But the corresponding comorphism

$$\mu^*\colon \mathbb{C}[y,z] \to \mathbb{C}[y,z],\ Q(y,z) \mapsto \overline{Q}(z,y)$$

does not preserve $\mathbb{C}[y, yz, z^2, z^3]$ (as μ^* exchanges y and z), and so μ does not extend to a real structure on X.

We do not know whether a similar phenomenon can occur when X_0 is a nilpotent orbit with non-normal closure. More precisely, by the results we have proven, we know that any (G, σ)-equivariant real structure on a nilpotent orbit is equivalent to one that extends (whether the closure of the orbit is normal or not), namely $d\sigma_e$. However, in the case where the nilpotent orbit does not have a normal closure, it is *a priori* possible that we have two equivalent real structures, one that extends, and one that does not, or even two that extend to inequivalent real structures.

6. Back to Example 1.2

In this last section, we justify that the map μ, given in Example 1.2, is a (G, σ)-equivariant real structure on $\mathcal{O}_{reg}$ that does not extend to a (G, σ)-equivariant real structure on $\mathfrak{sl}_3$. (Of course, it follows from our Main Theorem that μ is equivalent to a (G, σ)-equivariant real structure that does extend to $\mathfrak{g}$, namely $(d\sigma_e)_{|\mathcal{O}_{reg}}$, but we will see that this does not imply that μ itself extends to $\mathfrak{g}$.)

Recall that, in this example, $G = \mathrm{SL}_3(\mathbb{C})$, and $\sigma(g) = \bar{g}$ for all $g \in G$. To check that μ is indeed a (G, σ)-equivariant real structure on $\mathcal{O}_{reg}$, it suffices to check that the element $g_0 := \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfies the two conditions of Lemma 2.9 with

$$H := \mathrm{Stab}_G\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix};\ a,b,c \in \mathbb{C} \text{ with } a^3 = 1\right\},$$

which is a straightforward computation left to the reader.

Let $\mu_0 := (d\sigma_e)_{|\mathcal{O}_{reg}}$, and let $\varphi := \mu_0 \circ \mu \in \mathrm{Aut}_{\mathbb{C}}^G(\mathcal{O}_{reg})$. We check that

$$\varphi \colon \mathcal{O}_{reg} \to \mathcal{O}_{reg}, \; g \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \left(g \begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = g \cdot \begin{bmatrix} 0 & 1 & -i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since μ_0 is the restriction of $d\sigma_e \colon \mathfrak{sl}_3 \to \mathfrak{sl}_3$, we see that μ extends to a (G,σ)-equivariant real structure on $\mathfrak{sl}_3$ if and only if there exists $\Phi \in \mathrm{Aut}_{\mathbb{C}}^G(\mathfrak{sl}_3)$ such that $\Phi_{|\mathcal{O}_{reg}} = \varphi$.

Suppose that such a Φ exists. Then, for any $u \in \mathfrak{sl}_3$, the isotropy subgroups of u and $\Phi(u)$ are the same. Let $u_{a,b} := \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{bmatrix}$ with $a, b \in \mathbb{C}^*$. Then

$$\left\{ \begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{bmatrix} \; \middle| \; \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\} \text{ is the isotropy subgroup of } u_{a,b} \text{ in } G. \text{ Moreover,}$$

we check that $\{u_{a,b} \mid a, b \in \mathbb{C}^*\}$ are the only elements in $\mathfrak{sl}_3$ with this isotropy subgroup, and so this set must be stable by Φ.

Let f_1, f_2 be two $\mathbb{C}$-scheme endomorphisms of $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ such that $\Phi(u_{a,1}) = u_{f_1(a), f_2(a)}$. It is well-known that $f_i(a) = \nu_i a^{\epsilon_i}$ with $\nu_i \in \mathbb{C}^*$ and $\epsilon_i \in \mathbb{Z}$. Let $w_a := \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & -2a \end{bmatrix}$ and let $g_a := \frac{1}{3} \begin{bmatrix} a^{-1} & 0 & 3 \\ 0 & a^{-1} & -9a \\ 0 & 0 & 27a^2 \end{bmatrix}$. Then we check that $w_a = g_a \cdot u_{a,1}$, and thus

$$\begin{bmatrix} 0 & 1 & -i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \Phi \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \Phi \left(\lim_{a \to 0} w_a \right) = \lim_{a \to 0} \Phi(w_a) = \lim_{a \to 0} (g_a \cdot \Phi(u_{a,1}))$$

$$= \lim_{a \to 0} \left(g_a \cdot u_{f_1(a), f_2(a)} \right) = \lim_{a \to 0} \begin{bmatrix} \nu_1 a^{\epsilon_1} & \nu_2 a^{\epsilon_2} & \frac{1}{3}(\nu_2 a^{\epsilon_2 - 1} - \nu_1 a^{\epsilon_1 - 2}) \\ 0 & \nu_1 a^{\epsilon_1} & \nu_1 a^{\epsilon_1 - 1} \\ 0 & 0 & -2\nu_1 a^{\epsilon_1} \end{bmatrix}.$$

This implies readily that $\epsilon_1 = 1, \epsilon_2 = 0, \nu_1 = \nu_2 = 1$, but then $\nu_2 a^{\epsilon_2 - 1} - \nu_1 a^{\epsilon_1 - 2} = 0$, which is absurd. Hence, there is no $\Phi \in \mathrm{Aut}_{\mathbb{C}}^G(\mathfrak{sl}_3)$ such that $\Phi_{|\mathcal{O}_{reg}} = \varphi$, and so the (G,σ)-equivariant real structure μ given in Example 1.2 is not obtained by restricting a (G,σ)-equivariant real structure from $\mathfrak{sl}_3$ to $\mathcal{O}_{reg}$.

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