$\mathfrak{p}$-self-large elements of symmetric Lie algebras

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Abstract

This note is a translation of [Bu, Appendix]. Let $\mathfrak{g}$ be a semisimple Lie algebra. A nilpotent element e of $\mathfrak{g}$ is said to be *self-large* if each nilpotent element commuting with e belongs to the closure of the adjoint orbit of e. We study an analogue of this in the framework of symmetric Lie algebras. In particular, we give a complete classification of the so called $\mathfrak{p}$-self-large elements.

Let $\mathfrak{g}$ be a semisimple Lie algebra defined over an algebraically closed field of characteristic 0. Let G be its adjoint group and κ its Killing form. Let θ be an involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated decomposition into, respectively, $+1$-θ-eigenspace and -1-θ-eigenspace. We say that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric Lie algebra. We adopt the notation of [He] to refer to classes of simple symmetric Lie algebras. For example, there are 3 classes of symmetric Lie algebras such that $\mathfrak{g}$ is of type A: AI, AII and AIII. The correspondence between types and data $(\mathfrak{g}, \mathfrak{k})$ is recalled in table 1.

Define K as the connected subgroup of G with Lie algebra $\mathfrak{k}$. If $A \subset \mathfrak{g}$, we denote the set of nilpotent elements of A by $\mathcal{N}(A)$. When $\mathfrak{g}$ is exceptional, we use the numbering of K-nilpotent orbits of $\mathfrak{p}$ given in [Dj1, Dj2]. If $e \in \mathfrak{g}$, the centralizer of e in $\mathfrak{g}$ is denoted by $\mathfrak{g}^e := \{x \in \mathfrak{g} \mid [x, e] = 0\}$. If $A \subset \mathfrak{g}$, we define $\mathfrak{g}^A$ as $\cap_{e\in A}\mathfrak{g}^e$. If $e \in \mathfrak{p}$ is a non-zero nilpotent element, we embed e in a normal $\mathfrak{sl}_2$-triple (e, h, f), i.e. $h \in \mathfrak{k}$ and $f \in \mathfrak{p}$. This yield the characteristic grading for $\mathfrak{w} = \mathfrak{g}, \mathfrak{k}$ or $\mathfrak{p}$

$$\mathfrak{w} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{w}(i, h); \qquad \mathfrak{w}^e = \bigoplus_{i\geqslant 0} \mathfrak{w}(e, i); \qquad \mathfrak{w}^f = \bigoplus_{i\leqslant 0} \mathfrak{w}(f, i).$$

where $\mathfrak{w}(i, h) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$, $\mathfrak{w}(e, i) = \mathfrak{w}^e \cap \mathfrak{w}(i, h)$ and $\mathfrak{w}(f, i) = \mathfrak{w}^f \cap \mathfrak{w}(i, h)$.

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Definition 1. A non-zero nilpotent element $e \in \mathfrak{p}$ is said to be

- $\mathfrak{p}$-distinguished if $\mathfrak{w}(e,0) = \{0\}$.
- almost $\mathfrak{p}$-distinguished if $\mathfrak{w}(e,0)$ is a toral subalgebra of $\mathfrak{g}$.
- $\mathfrak{p}$-self-large if $\mathcal{N}(\mathfrak{p}^e) \subset \overline{K.e}$.

In fact, an element $e \in \mathfrak{p}$ is $\mathfrak{p}$-distinguished if and only if $\mathfrak{p}^e \subset \mathcal{N}(\mathfrak{p})$ if and only if $\mathfrak{p}^e \subset \overline{K.e}$, cf. [Bu, §1.3&1.4]. In [Bu, Lemme 1.8] it is also shown that only $\mathfrak{p}$-self-large elements are likely to span an irreducible component of $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) := \{(x,y) \in \mathfrak{p} \times \mathfrak{p} \mid [x,y] = 0\}$, the nilpotent commuting variety of $\mathfrak{p}$. In addition, the following implications are proved.

$$\mathfrak{p}\text{-distinguished} \Rightarrow \mathfrak{p}\text{-self-large} \Rightarrow \text{almost } \mathfrak{p}\text{-distinguished}.$$

The almost $\mathfrak{p}$-distinguished elements of $\mathfrak{p}$ are easily classified from the knowledge of the characteristic grading, cf. [Bu]. We aim for a criterion which characterizes almost $\mathfrak{p}$-distinguished elements which are not $\mathfrak{p}$-self-large. From now on, we assume that e is almost $\mathfrak{p}$-distinguished. By definition, $\mathfrak{p}(e,0)$ is a toral subalgebra of $\mathfrak{g}$. It acts on $\mathfrak{g}(f,-1)$ via the adjoint action of $\mathfrak{g}$. Let $\mathfrak{X}(\mathfrak{p}(e,0)) \subset (\mathfrak{p}(e,0))^*$ be the set of weights of this representation. The weight decomposition will be denoted by

$$\mathfrak{g}(f,-1) = \bigoplus_{\gamma \in \mathfrak{X}(\mathfrak{p}(e,0))} V_\gamma.$$

The purpose of the forthcoming proposition 4 is to give a weak analogue of [Pa, Theorem 2.1] for symmetric Lie algebras. It turns out that, combined with computations of [Bu], this proposition is sufficient for obtaining a complete list of $\mathfrak{p}$-self-large orbits in each symmetric Lie algebras. First, we need the following lemmas.

Lemma 2 (Panyushev [Pa])**.** *The map*

$$\Phi : \left\{ \begin{array}{rcl} \mathfrak{g}(f,-1) \times \mathfrak{g}(f,-1) & \to & \Bbbk \\ (\xi,\eta) & \mapsto & \kappa(e,[\xi,\eta]) \end{array} \right.$$

is a non-degenerate skew-symmetric $\mathfrak{g}(e,0)$-invariant bilinear form.

Lemma 3. *The automorphism θ induces a bijection between V_γ and $V-\gamma$. The application $\tilde{\Phi} : (\xi,\eta) \mapsto \Phi(\xi,\theta(\eta))$ is a non-degenerate symmetric bilinear form. It remains non-degenerate on each subspace V_γ*

Proof. Since $h \in \mathfrak{k}$ and $f \in \mathfrak{p}$, the automorphism θ induces an automorphism of $\mathfrak{g}(f, -1)$. Let ξ be an element of V_γ. For all $t \in \mathfrak{p}(e, 0)$, we have

$$[t, \theta(\xi)] = \theta([\theta(t), \xi]) = \theta(-\gamma(t)\xi) = -\gamma(t)\theta(\xi)$$

which proves the first assertion.

Since θ induces an automorphism of $\mathfrak{g}(f, -1)$, the non-degeneracy of $\tilde{\Phi}$ is a consequence of the non-degeneracy of Φ. Let us check that $\tilde{\Phi}$ is symmetric.

$$\begin{aligned} \kappa(e, [\xi, \theta(\eta)]) &= \kappa(\theta(e), \theta([\xi, \theta(\eta)])) \\ &= \kappa(-e, [\theta(\xi), \eta]) \\ &= \kappa(e, [\eta, \theta(\xi)]). \end{aligned}$$

Let $\xi \in V_\gamma$, $\eta \in V_\mu$ and $t \in \mathfrak{p}(e, 0)$, then $\theta(\eta) \in V_{-\mu}$ and for all $t \in \mathfrak{p}(e, 0)$, we have

$$(\gamma(t) - \mu(t))\tilde{\Phi}(\xi, \eta) = \Phi([t, \xi], \theta(\eta)) + \Phi(\xi, [t, \theta(\eta)]) = 0.$$

In particular, $\tilde{\Phi}(\xi, \eta) = 0$ if $\gamma \neq \mu$. This proves the non-degeneracy of $\tilde{\Phi}$ on V_γ. $\square$

Proposition 4. *Assume that e is almost $\mathfrak{p}$-distinguished, $\mathfrak{g}(f, -1)^{\mathfrak{p}(e,0)} = \{0\}$ and $\mathfrak{p}(e, 1) \neq \{0\}$. Then e is not $\mathfrak{p}$-self-large .*

Proof. We have $0 \notin \mathfrak{X}(\mathfrak{p}(e, 0))$ from the hypothesis. Fix $\mu \in \mathfrak{X}(\mathfrak{p}(e, 0))$ and let $\xi \in V_\mu$ such that $\tilde{\Phi}(\xi, \xi) \neq 0$. Since $\mu \neq 0$, there exists $t \in \mathfrak{p}(e, 0)$ such that $[t, \xi] = \xi$ and $[t, \theta(\xi)] = -\theta(\xi)$. An easy computation gives

$$\kappa([[e, (\xi + \theta(\xi))], (\xi + \theta(\xi))], t) = 2\kappa(e, [\xi, \theta(\xi)]) = 2\tilde{\Phi}(\xi, \xi) \neq 0.$$

Hence $z = \xi + \theta(\xi)$ is an element of $\mathfrak{k}(f, -1)$ satisfying $[[e, z], z] \neq 0$. Finally, thanks to [Pa, Lemma 2.3], we know that $G.(e + [z, e])$ is strictly larger orbit than $G.e$ and, in particular, $K.(e + [z, e]) \not\subseteq \overline{K.e}$. On the other hand, $[z, e] \in \mathfrak{p}(e, 1)$ hence $e + [z, e] \in \mathfrak{p}^e$. $\square$

Remark 5. We can not drop the hypothesis $\mathfrak{g}(f, -1)^{\mathfrak{p}(e,0)} = \{0\}$. Indeed, if $e \in \mathcal{O}_1$ in FII, it satisfies $\mathfrak{p}(e, 0) = \{0\}$ and $\mathfrak{p}(e, 1) \neq \{0\}$. However, e is $\mathfrak{p}$-distinguished, and therefore $\mathfrak{p}$-self-large.

Corollary 6. *The orbits $\mathcal{O}_{50}$ of EV, $\mathcal{O}_{85}$, $\mathcal{O}_{88}$ of EVIII, $\mathcal{O}_{16}$, $\mathcal{O}_{17}$ of EI are not $\mathfrak{p}$-self-large.*

Proof. We know from [Bu, Section 6] that these nilpotent orbits are almost $\mathfrak{p}$-distinguished orbits and are not even. Using [Dj1, Dj2], we can compute $\mathfrak{p}(e,0)$ and we find that $\mathfrak{p}(e,0) = \mathfrak{g}(e,0)$ in these cases.

Now the following argument taken from [Pa] applies. Let $\mathfrak{l} = \mathfrak{g}^{\mathfrak{g}(e,0)}$ and $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$. The Levi factor $\mathfrak{l}$ can be decomposed in $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{g}(e,0)$. Then e is distinguished in $\mathfrak{l}$, and the grading induced by $h \in \mathfrak{l}$ satisfies

$$0 = \mathfrak{s}(-1,h) = \mathfrak{l}(-1,h) = \mathfrak{g}(-1,h)^{\mathfrak{g}(e,0)} = \mathfrak{g}(-1,h)^{\mathfrak{p}(e,0)} \supseteq \mathfrak{g}(f,-1)^{\mathfrak{p}(e,0)}.$$

Combining this with the tables of [JN], we can show that the hypothesis of proposition 4 are satisfied and so, the orbits considered here are not $\mathfrak{p}$-self-large. □

Following the same idea, it is possible to give an alternative way to [Bu, Section 5.2] to find almost $\mathfrak{p}$-distinguished orbits of AI which are not $\mathfrak{p}$-self-large. Unfortunately, the proposition 4 can not be applied to almost $\mathfrak{p}$-distinguished orbits of AII and $\mathcal{O}_1$ of EIV. For these orbits we refer to [Bu, Sections 5.3, 6.3]. Almost $\mathfrak{p}$-distinguished orbits being classified in [Bu], we are now able to list all $\mathfrak{p}$-self-large orbits in the different simple cases. The general case easily follows, since symmetric Lie algebras are direct products of simple symmetric Lie algebras.

Table 1: List of $\mathfrak{p}$-self large orbits

Type	$(\mathfrak{g}, \mathfrak{k})$	$\mathfrak{p}$-self-large orbits.
AI	$(\mathfrak{sl}_n, \mathfrak{so}_n)$	The ($\mathfrak{p}$-distinguished) regular orbit and the orbits whose associated partition have only values differing by at least 2.
AII	$(\mathfrak{sl}_n, \mathfrak{sp}_n)$	The ($\mathfrak{p}$-distinguished) regular orbit and the orbits whose associated doubled partition have only pairs whose values differ by at least 2.
AIII	$(\mathfrak{sl}_n, \mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{k})$	The $\mathfrak{p}$-distinguished orbits (i.e. that have an ab-diagram whose lines of same length begin by the same letter). They are the only almost $\mathfrak{p}$-distinguished orbits.
BDI, CI	$(\mathfrak{so}_n, \mathfrak{so}_p \oplus \mathfrak{so}_q)$, $(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$	The almost $\mathfrak{p}$-distinguished orbits (cf. [Bu, Section 3.3]).
CII, DIII	$(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q})$, $(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$	The $\mathfrak{p}$-distinguished orbits (which are also the only almost $\mathfrak{p}$-distinguished orbits, cf. [Bu, Section 3.3]).

EIII, EVI, EVII, EIX, FI, FII, GI	$(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{k})$, $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$ $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{k})$, $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{sl}_2)$, $(\mathfrak{f}_4, \mathfrak{sp}_6 \oplus \mathfrak{sl}_2)$, $(\mathfrak{f}_4, \mathfrak{so}_9)$	The $\mathfrak{p}$-distinguished orbits (which are also the only almost $\mathfrak{p}$-distinguished orbits, cf. [Bu, Section 6.1]).
EII	$(\mathfrak{e}_6, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2)$	The almost $\mathfrak{p}$-distinguished orbits. In particular the orbit $\mathcal{O}_{22}$ which is not $\mathfrak{p}$-distinguished.
EIV	$(\mathfrak{e}_6, \mathfrak{f}_4)$	The regular ($\mathfrak{p}$-distinguished) orbit.
EI	$(\mathfrak{e}_6, \mathfrak{sp}_8)$	The $\mathfrak{p}$-distinguished orbits and $\mathcal{O}_{12}$, $\mathcal{O}_{21}$, $\mathcal{O}_{23}$ (cf. [Bu, Section 6 and Lemma 1.9]).
EV	$(\mathfrak{e}_7, \mathfrak{sl}_9)$	The $\mathfrak{p}$-distinguished orbits and $\mathcal{O}_{81}$ (cf. [Bu, Section 6 and Lemma 1.9]).
EVIII	$(\mathfrak{e}_8, \mathfrak{so}_{16})$	The $\mathfrak{p}$-distinguished orbits and $\mathcal{O}_{81}$, $\mathcal{O}_{95}$ (cf [Bu, Section 6 and Lemma 1.9]).
Lie algebras		Orbits of elements e for which $\mathfrak{p}(e,0)$ is a torus and $\mathfrak{p}(e,1) = 0$ (cf. [Pa]).

Furthermore, the computations yield to the following remark. The $\mathfrak{p}$-self-large elements in a simple symmetric Lie algebra are exactly the elements e which satisfy at least one of the following two conditions

- $\mathfrak{p}(e,0) = \{0\}$ i.e. e is $\mathfrak{p}$-distinguished.
- $\mathfrak{p}(e,0)$ is a torus and $\mathfrak{p}(e,1) = \{0\}$.

Note that this is only the case for simple algebras. Indeed, Remark 5 implies that in FII×EI, the orbit $\mathcal{O}_1 \times \mathcal{O}_{21}$ is $\mathfrak{p}$-self-large, satisfies $\mathfrak{p}(e,0) \cong \mathfrak{t}_1 \neq \{0\}$ and $\mathfrak{p}(e,1) \neq \{0\}$.

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