Answer to arxiv:1401.1532 and link with the $3x+1$ conjecture

Michaël Bulois[†], Frédéric Chardard[†], Olivier Marchal[†]

Abstract

In [Ze], Doron Zeilberger conjectures that a combinatorial identity involving a determinant should hold. We show in this note that this identity holds if and only if there is no cycle in the $3x+1$ problem. We only use elementary techniques. That is not a proof of the conjecture of [Ze], but hey, that is worth 500$ for OEIS.

1 Statement of the conjectures

Let d be a positive integer, and Let $M = M(d)$ be the following $2d \times 2d$ matrix with entries in $\{-1, 0, 1\}$. For $1 \leqslant a \leqslant 2d$ and $1 \leqslant b \leqslant d$,

$$M_{a,2b-1} = \begin{cases} 1 & \text{if } a = 2b; \\ -1 & \text{if } a = 3b+1; \\ 0 & \text{otherwise} \end{cases}$$

$$M_{a,2b} = \begin{cases} 1 & \text{if } a = 2b-1; \\ -1 & \text{if } a = b-1; \\ 0 & \text{otherwise} \end{cases}$$

In [Ze], the following is conjectured:

Conjecture 1. *For every positive integer d, we have*

$$\det\left(M(d)\right) = (-1)^d.$$

On the other hand, we remind the reader of the $3x+1$ conjecture. Define a function $f : \mathbb{N}^* \to \mathbb{N}^*$ via

$$f(n) := \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

[†] Université de Lyon, CNRS UMR 5208, Université Jean Monnet, Institut Camille Jordan, France
michael.bulois@univ-st-etienne.fr frederic.chardard@univ-st-etienne.fr
olivier.marchal@univ-st-etienne.fr

The $3x+1$ conjecture, also known as Collatz conjecture or Syracuse problem, can be stated as the following.

Conjecture 2 (see e.g. [Wi])**.** *For any $n \in \mathbb{N}^*$, there exists $k \in \mathbb{N}^*$ such that $f^{(k)}(n) = 1$.*

There is a weaker (but still very open) version of this conjecture which states that there are no trivial cycle induced by the function f.

Conjecture 3. *Let $n \in \mathbb{N}^* \setminus \{1,2\}$, then*

$$\forall k \in \mathbb{N}^*, f^{(k)}(n) \neq n.$$

The last version is weaker because it does not exclude that iterating the f function will not create an unbounded serie. We will show in this short note that Conjectures 1 and 3 are equivalent.

2 Equivalence

Let $M'(d)$ be the matrix obtained from $M(d)$ by switching the columns of index $2b-1$ with those of index $2b$. That is

$$M'_{a,2b} := M'_{a,2b-1}, \qquad M'_{a,2b-1} := M_{a,2b}.$$

Since we exchanged d pairs of columns, conjecture 1 becomes:

$$\det\left(M'(d)\right) = 1. \tag{1}$$

We claim that it is simpler to study M' than M. Note in particular that for any index j, we have $M'_{j,j} = 1$. Set

$$N := \mathrm{Id} - M'$$

where Id is the $2d \times 2d$ identity matrix. From the definition we have:

$$\begin{cases} N_{i,2i+1} & = 1\,,\ \forall i \in [\![1, d-1]\!] \\ N_{3i+1,2i} & = 1\,,\ \forall\, i \in [\![1, E\left(\frac{2d-1}{3}\right)]\!] \\ N_{i,j} & = 0\ ,\ \text{otherwise} \end{cases} \tag{2}$$

For example we have for $d = 6$:

$$N(6) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In other words, define the function $g : \mathbb{N} \to \mathbb{N}$ by the formula:

$$g(j) := \begin{cases} \frac{3}{2}j + 1 & \text{if } j \text{ is even} \\ \frac{j-1}{2} & \text{if } j \text{ is odd.} \end{cases}$$

We see that, for each $j \in [\![1, 2d]\!]$, there exists at most one index $i \in [\![1, 2d]\!]$ such that $N_{i,j} \neq 0$. This index exists if and only if $g(j) \in [\![1, 2d]\!]$ and satisfies $i = g(j)$. We then have the following lemma:

Lemma 4. *N is a nilpotent matrix if and only if for any $j \in [\![1, 2d]\!]$, there exists $k \in \mathbb{N}^*$ such that either*

(a) $g^{(k)}(j) < 1$ (hence $g^{(k)}(j) = 0$)

(b) $g^{(k)}(j) > 2d$.

Otherwise, 1 is as an eigenvalue of N.

Proof. If conditions (a) or (b) are satisfied for some k, j, then N^k maps e_j to 0, where e_j is the j-th vector of the canonical basis. If they are true for any j then N is nilpotent.
On the other hand, if there exists j such that neither (a) nor (b) are satisfied, N induces a cycle $e_{j_0} \to e_{j_1} \to \cdots \to e_{j_\ell} = e_{j_0}$ with $j_0 = j$ and $j_{m+1} = g(j_m)\ \forall m \in [\![0, l-1]\!]$. Then $v := \sum_{m=0}^{\ell-1} e_{j_m}$ satisfies $N.v = v$ and thus 1 is an eigenvalue of N and N is non-nilpotent. ☐

Now, we are now able to prove the following.

Theorem 5. *Conjecture 1 is equivalent to Conjecture 3.*

Proof. An easy induction yields for $k \in \mathbb{N}^*$ that

$$g^{(k)}(j) + 1 = f^{(k)}(j + 1). \tag{3}$$

In particular, cycles in $\mathbb{N}$ for g corresponds to cycles in $\mathbb{N}^*$ for f. That is, Conjecture 3 holds if and only if there are no cycle in $\mathbb{N}$ for g, apart the trivial one involving $1, 0$ (corresponding to the cycle $1, 2$ for the usual Syracuse f problem).

Now, it follows from Lemma 4 that any non-trivial cycle for g yields a non-nilpotent matrix N with 1 as an eigenvalue (taking $2d$ sufficiently large). Since $M' = Id - N$, this would yield a non-trivial kernel for M' and we would have $\det M' = 0$ which contradicts (1).

On the other hand, the non-existence of such cycle would imply $N^{2d} e_j = 0$ $\forall j \in [\![1, 2d]\!]$ hence the nilpotency of N. In particular, M' would have 1 as only eigenvalue hence (1) is satisfied. □

Remark 6. We have even shown that whenever $\det(M(d_0)) \neq (-1)^{d_0}$ (or, equivalently, there exists a non-trival cycle with greater value less or equal to $2d_0$), then $\det(M(d)) = 0$ for all $d \geqslant d_0$.

3 Conclusion

We have shown here that Conjectures 1 and 3 are equivalent. Since general opinion is that there are no other cycle than $\{1, 2\}$ in the Syracuse problem, it seems that 1 holds. Both problems seeming of equivalent difficulty, it is very unlikely that some elementary determinant techniques may prove simply 1. However, just in the case that someone have an proof of 1, we are volunteer to co-author a proof of $3x + 1$ conjecture :-) .

References

[Wi] Wikipedia, page on Collatz conjecture, version of January 9, 2014..

[Ze] Doron Zeilberger, An Explicit Conjectured Determinant Evaluation Whose Proof Would Make Me Happy (and the OEIS richer), arxiv:1401.1532.