Fast computation of the Maslov index for hyperbolic linear systems with periodic coefficients

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Abstract

The Maslov index is a topological property of periodic orbits of finite-dimensional Hamiltonian systems that is widely used in semiclassical quantization, quantum chaology, stability of waves and classical mechanics. The Maslov index is determined from analysis of a linear Hamiltonian system with periodic coefficients. In this paper a numerical scheme is devised to compute the Maslov index for *hyperbolic* linear systems when the phase space has low dimension. The idea is to compute on the exterior algebra of the ambient vector space, where the Lagrangian subspace representing the unstable subspace is reduced to a line. When the exterior algebra is projectified the Lagrangian subspace always forms a closed loop. The idea is illustrated by application to Hamiltonian systems on a phase space of dimension four. The theory is used to compute the Maslov index for the spectral problem associated with periodic solutions of the fifth-order KdV equation.

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1 Introduction

The Maslov index is a property of periodic orbits of Hamiltonian systems that is required in a wide range of physical applications: semiclassical quantization [2, 9, 16, 15], quantum chaology [10], and classical mechanics [4]. The linearization about a periodic orbit leads to a Hamiltonian system with periodic coefficients. In this paper, with the assumption that the linear system is hyperbolic, the Maslov index can be interpreted as a winding number of a family of Lagrangian planes of the linear Hamiltonian system.

A linear Hamiltonian system with $T-$periodic coefficients can be written in the standard form

$$\mathbf{J}\mathbf{z}_t = \mathbf{B}(t)\mathbf{z}\,, \quad \mathbf{z} \in \mathbb{R}^{2n}\,, \quad \mathbf{B}(t+T) = \mathbf{B}(t)\,, \quad t \in \mathbb{R}\,, \tag{1.1}$$

where $\mathbf{B}(t)$ is a symmetric matrix and

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} . \tag{1.2}$$

Hyperbolic linear systems have the property that the unstable subspace is Lagrangian [16, 17]. A T-periodic linear system is said to be hyperbolic if the monodromy operator $\mathbf{M} := \mathbf{\Phi}(t_0 + T, t_0)$ has no eigenvalue with a modulus equal to one, where $\mathbf{\Phi}(t, t_0)$ is the fundamental solution matrix of (1.1) defined by

$$\mathbf{J}\mathbf{\Phi}_t = \mathbf{B}(t)\mathbf{\Phi}\,, \quad \mathbf{\Phi}(t_0, t_0) = \mathbf{I}\,, \quad t \geq t_0\,.$$

Let $\{\zeta_1, \ldots, \zeta_n\}$ be a basis associated with the Floquet multipliers of $\mathbf{M}$ with modulus greater than one. Then $\{\mathbf{\Phi}(t, t_0)\zeta_1, \ldots, \mathbf{\Phi}(t, t_0)\zeta_n\}$ provides a basis for the unstable subspace at each t. The subspace span$\{\zeta_1, \ldots, \zeta_n\}$ is Lagrangian, and span$\{\mathbf{\Phi}(\cdot, t_0)\zeta_1, \ldots, \mathbf{\Phi}(\cdot, t_0)\zeta_n\}$ is a path of Lagrangian subspaces.

A subspace, span$\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset \mathbb{R}^{2n}$, is a Lagrangian subspace if it has dimension n and $\langle \mathbf{J}\mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for each $\mathbf{u}_i, \mathbf{u}_j$. The set of all Lagrangian subspaces is a manifold of dimension $\frac{1}{2}n(n+1)$ and is denoted by Λ_n. Every closed path of Lagrangian subspaces has a Maslov index which is integer valued [2]. A point $\mathbf{U} \in \Lambda_n$ can be represented by a $2n \times n$ matrix $\begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ matrices satisfying $\mathbf{A}_1^T\mathbf{A}_2 = \mathbf{A}_2^T\mathbf{A}_1$. Lagrangian subspaces are invariant under the flow of (1.1). Hence, $\mathbf{U}(t)$ satisfying

$$\mathbf{J}\mathbf{U}_t = \mathbf{B}(t)\mathbf{U}\,, \quad \mathbf{U}(t_0) \in \Lambda_n\,, \quad t \geq t_0\,, \tag{1.3}$$

defines a path of Lagrangian subspaces, since

$$\frac{d}{dt}\mathbf{U}(t)^T\mathbf{J}\mathbf{U}(t) = -\mathbf{U}(t)^T\mathbf{B}(t)\mathbf{U}(t) + \mathbf{U}(t)^T\mathbf{B}(t)\mathbf{U}(t) = 0\,,$$

and so $\mathbf{U}(t)^T\mathbf{J}\mathbf{U}(t) = \mathbf{U}(t_0)^T\mathbf{J}\mathbf{U}(t_0) = 0$ when $\mathbf{U}(t_0) \in \Lambda_n$. Given a path of Lagrangian subspaces, define the *Maslov angle* $\kappa(t)$ by

$$\mathrm{e}^{\mathrm{i}\kappa(t)} = \frac{\det(\mathbf{A}_1(t) - \mathrm{i}\mathbf{A}_2(t))}{\det(\mathbf{A}_1(t) + \mathrm{i}\mathbf{A}_2(t))}\,, \quad \mathbf{U}(t) = \begin{pmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{pmatrix} . \tag{1.4}$$

When the path of Lagrangian subspaces is closed, say $\mathbf{U}(t_0 + T) = \mathbf{U}(t_0)$, the *Maslov index* is clearly an integer defined by

$$m = \frac{\kappa(T) - \kappa(0)}{2\pi} . \tag{1.5}$$

However, for systems of the type (1.1) the loop does not close but $\mathbf{U}(t_0+T) = \mathbf{U}(t_0)\mathbf{P}(t_0)$ for some $n \times n$ matrix $\mathbf{P}(t_0)$. But the determinant of $\mathbf{P}(t_0)$ cancels out in the formula for the Maslov angle. In other words, *the path of the unstable Lagrangian subspace induced on projectified* $\bigwedge^n(\mathbb{R}^{2n})$ *does form a closed loop* (see §3). Hence the formula (1.5) still leads to an integer. The hypotheses of hyperbolicity and projectification are both essential.

Although the angle $\kappa(t)$ in (1.4) is easy to define, its computation can be cumbersome. In principle, the Maslov index can be evaluated by computing the Floquet multipliers and their eigenvectors, and then taking the n eigenvectors associated with the unstable subspace of (1.1) as initial conditions. The ODE (1.1) can then in principle be integrated from $t = 0$ to $t = T$ and the angle $\kappa(t)$ computed as a function of t [14, 15].

There are two problems with this approach to computing the Maslov index: the individual Floquet multipliers need to be computed along with a basis for the unstable subspace; secondly, the integration of individual vectors over an interval $[t_0, t_0 + T]$ will not be stable as numerically the solutions will always be attracted to the most unstable direction. To solve the latter problem some stabilization is required such as orthogonalization, but discrete orthogonalization increases computation time and continuous orthogonalization transforms the ODE to a nonlinear problem [7].

When the dimension of the phase space is not too large, these problems can be overcome by integrating the system (1.1) restricted to the exterior algebra space $\bigwedge^n(\mathbb{R}^{2n})$. On the exterior algebra space, the $n-$dimensional unstable subspace is transformed to a line. This line represents the entire unstable subspace and so is globally attracting. Therefore the system on $\bigwedge^n(\mathbb{R}^{2n})$ can be integrated with a random initial condition. No explicit computation of Floquet multipliers is required.

Exterior algebra has been widely used for integrating non-Hamiltonian systems on $k-$dimensional subspaces (e.g. [1] and references therein), and the exterior algebra representation of the Maslov index has been used in the stability analysis of periodic and solitary waves [12, 5]. However, numerical computation of the Maslov index on exterior algebra spaces appears to be new.

The motivation of the authors is the application of the Maslov index to the stability of waves [12, 5, 8]. In this case, there is an external parameter (the spectral parameter; see §4), and for most values of this parameter the linear system is hyperbolic.

A related problem of interest is the Maslov index of a $T-$periodic orbit, $\widehat{\mathbf{z}}(t)$, of an autonomous Hamiltonian system,

$$\mathbf{J}\frac{d}{dt}\widehat{\mathbf{z}}(t) = \nabla H\left(\widehat{\mathbf{z}}\right) , \quad \widehat{\mathbf{z}} \in \mathbb{R}^{2n} . \tag{1.6}$$

In this case the linearization about $\widehat{\mathbf{z}}(t)$ takes the form (1.1) with $\mathbf{B}(t) = \mathrm{D}^2 H(\widehat{\mathbf{z}}(t))$ and there are always two Floquet multipliers at plus one, so the system can not be hyperbolic. When the other $2n - 2$ Floquet multipliers are hyperbolic the treatment of the Maslov index is similar to the hyperbolic case but the two unit modulus eigenvalues need to be accounted for. Some comments on this case are given in §5.

In this paper the details of the numerical method on exterior algebra spaces is developed for the case $n = 2$. The induced systems are linear, and any standard integrator can be used, leading to an elementary algorithm. For the case $n = 2$ we also find a new formulation of the induced system on $\bigwedge^2(\mathbb{R}^4)$ which closely matches the fibre-bundle structure of the Lagrangian Grassmannian. The algorithm is applied to the eigenvalue problem which arises in the linearization of the fifth-order KdV equation about periodic travelling waves.

2 The Maslov index on exterior algebra spaces

In this section the theory for the Maslov angle on exterior algebra spaces is developed for the case when the phase space has dimension 4. The restriction of system (1.1) to $\bigwedge^2(\mathbb{R}^4)$, which has dimension six, is obtained by taking a basis for $\bigwedge^2(\mathbb{R}^4)$ and then representing the matrix $\mathbf{J}^{-1}\mathbf{B}(t)$ with respect to this basis [1]. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis for $\mathbb{R}^4$. Then the standard induced basis is

$$\bigwedge^2(\mathbb{R}^4) = \mathrm{span}\,\{\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_3,\ \mathbf{e}_1 \wedge \mathbf{e}_4,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_4,\ \mathbf{e}_3 \wedge \mathbf{e}_4\}\ .$$

Let $\mathbf{A}^{(2)}(t) := \bigwedge^2(\mathbf{J}^{-1}\mathbf{B}(t))$ be the matrix representation of the induced matrix on $\bigwedge^2(\mathbb{R}^4)$ with respect to the standard basis (a formula for $\mathbf{A}^{(2)}$ is given in [1]), then the induced system is

$$U_t = \mathbf{A}^{(2)}(t)U\,, \quad U \in \bigwedge^2(\mathbb{R}^4) \cong \mathbb{R}^6\,. \tag{2.7}$$

The *Lagrangian Grassmannian* Λ_2 is the manifold of two-dimensional Lagrangian subspaces of $\mathbb{R}^4$. On $\bigwedge^2(\mathbb{R}^4)$ it has the representation [12, 5]

$$\Lambda_2 = \mathbb{P}(\{\ U \in \bigwedge^2(\mathbb{R}^4)\ :\ U_1U_6 - U_2U_5 + U_3U_4 = 0 \quad \text{and} \quad U_3 + U_4 = 0\ \})\,,$$

where $\mathbb{P}(\cdot)$ means the induced subset of projective space. The Lagrangian Grassmannian Λ_2 is a three-dimensional submanifold of the five-dimensional projective space $\mathbb{P}(\bigwedge^2(\mathbb{R}^4))$, and it is also a fibre bundle. The base manifold is S^1, the fibre is S^2 and the total space is Λ_2 [3].

Introduce new coordinates which follow closely the fibre bundle structure. Consider the following new basis for $\bigwedge^2(\mathbb{R}^4)$:

$$\begin{array}{rclrcl}
\widetilde{\mathbf{e}}_1 &=& \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4) & \widetilde{\mathbf{e}}_4 &=& \frac{1}{\sqrt{2}}(\ \mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3) \\
\widetilde{\mathbf{e}}_2 &=& \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4) & \widetilde{\mathbf{e}}_5 &=& \frac{1}{\sqrt{2}}(-\mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4) \\
\widetilde{\mathbf{e}}_3 &=& \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3) & \widetilde{\mathbf{e}}_6 &=& \frac{1}{\sqrt{2}}(\ \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4)
\end{array}\ . \tag{2.8}$$

Take coordinates $\mathbf{u} = (u_1, u_2)$, u_3 and $\mathbf{v} = (v_1, v_2, v_3)$ relative to this basis. Then $\dot{u}_3 = 0$ (since $u_3 = 0$ defines the Lagrangian plane in these coordinates), and the five coordinates $(\mathbf{u}, \mathbf{v})$ satisfy

$$\begin{array}{rcl}
\mathbf{u}_t &=& \mathbf{S}_1(t)\mathbf{u} + \mathbf{N}(t)\mathbf{v} \\
\mathbf{v}_t &=& \mathbf{S}_2(t)\mathbf{v} + \mathbf{N}(t)^T\mathbf{u}\,.
\end{array} \tag{2.9}$$

The matrices $\mathbf{S}_1(t)$ and $\mathbf{S}_2(t)$ are skew-symmetric and are computed to be

$$\mathbf{S}_1(t) = \mathrm{Tr}(\mathbf{B}(t)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,$$

and

$$\mathbf{S}_2(t) = \begin{pmatrix} 0 & 2b_{23} - 2b_{14} & b_{11} - b_{22} + b_{33} - b_{44} \\ -2b_{23} + 2b_{14} & 0 & 2b_{34} + 2b_{12} \\ -b_{11} + b_{22} - b_{33} + b_{44} & -2b_{34} - 2b_{12} & 0 \end{pmatrix}\,,$$

where b_{ij} represent the entries of $\mathbf{B}(t)$ in (1.1) and

$$\mathbf{N}(t) = \begin{pmatrix} b_{11} - b_{22} - b_{33} + b_{44} & -2b_{34} + 2b_{12} & 2b_{13} + 2b_{24} \\ 2b_{13} - 2b_{24} & 2b_{23} + 2b_{14} & -b_{11} - b_{22} + b_{33} + b_{44} \end{pmatrix}\,.$$

In these coordinates, the Lagrangian Grassmannian is defined by

$$\Lambda_2 = \mathbb{P}\left(\{(\mathbf{u},\mathbf{v}) \ : \ \mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = 0\}\right) .$$

In these coordinates the flow on the base manifold S^1 of Λ_2 is given by $\mathbf{u}(t)/\|\mathbf{u}(t)\|$, and the flow on the fibre S^2 is given by $\mathbf{v}(t)/\|\mathbf{v}(t)\|$. Since Λ_2 is a fibre bundle the flow on these two manifolds is globally connected via the differential equation (2.9).

The nice feature of the representation (2.9) is that the Maslov index is obtained from twice the winding number of the vector $\mathbf{u}$. Let

$$\theta(t) = \tan^{-1}\left(\frac{-u_2(t)}{u_1(t)}\right) ;$$

then the winding number of the vector $\mathbf{u}(t)$ is just $\frac{\theta(T)-\theta(0)}{2\pi}$. The Maslov angle $\kappa(t)$ is *twice* this winding number: $\kappa(t) = 2\theta(t)$ and so

$$\mathrm{e}^{\mathrm{i}\kappa(t)} = \frac{u_1(t) - \mathrm{i}u_2(t)}{u_1(t) + \mathrm{i}u_2(t)} , \tag{2.10}$$

with the Maslov index given by (1.5). A verification of this formula follows by starting with (1.4), transforming using Plücker coordinates [5, 12] and then applying the transformation (2.8). A detailed derivation of this result and its generalization to higher dimension is given in [8].

3 Floquet theory on $\bigwedge^n(\mathbb{R}^{2n})$

There is an induced Floquet theory on the exterior algebra space. For a hyperbolic linear system (1.1) there are n Floquet multipliers with modulus greater than one, and n with modulus less than one.

Choose $t_0 \in \mathbb{R}$ and define the monodromy matrix $\mathbf{M} = \mathbf{\Phi}(t_0+T, t_0)$. The monodromy matrix is dependent on t_0 but choosing any other starting value for t_0, say t_1, leads to a monodromy operator $\widetilde{\mathbf{M}} = \mathbf{\Phi}(t_1+T, t_1)$ which has the same eigenvalues. The eigenvalues of any $\mathbf{M}$ in this equivalence class are the Floquet multipliers.

Denote by $\mathbf{M}^{(n)}$ the operator induced on $\bigwedge^n(\mathbb{R}^{2n})$ defined by

$$\mathbf{M}^{(n)}(Y_1\wedge\cdots\wedge Y_n) = \mathbf{M}Y_1\wedge\cdots\wedge\mathbf{M}Y_n , \quad \forall \quad Y_1,\ldots,Y_n \in \mathbb{R}^{2n} .$$

If $\mu_1,\ldots,\mu_n$ are any n Floquet multipliers with eigenvectors $\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_n$, then clearly

$$\mathbf{M}^{(n)}(\boldsymbol{\zeta}_1\wedge\cdots\wedge\boldsymbol{\zeta}_n) = \sigma\,(\boldsymbol{\zeta}_1\wedge\cdots\wedge\boldsymbol{\zeta}_n) , \quad \text{with} \quad \sigma = \prod_{j=1}^{n}\mu_j .$$

Since the system is hyperbolic, there is a unique simple Floquet multiplier of largest modulus of $\mathbf{M}^{(n)}$ obtained by taking $\{\zeta_1,\ldots,\zeta_n\}$ to be a basis for the unstable subspace. Denote this Floquet multiplier by σ_+. It is always simple and real, even if some of the Floquet multipliers are complex.

Consider the induced system

$$U_t = \mathbf{A}^{(n)}(t)U , \quad U(t_0) = \boldsymbol{\zeta}_1\wedge\cdots\wedge\boldsymbol{\zeta}_n ,$$

with $\{\zeta_1, \ldots, \zeta_n\}$ a basis for the unstable subspace. Then

$$U(t_0 + T) = \sigma_+ (\boldsymbol{\zeta}_1 \wedge \cdots \wedge \boldsymbol{\zeta}_n) ;$$

that is, $U(t_0 + T)$ and $U(t_0)$ are collinear. Hence, $U(t + T) = U(t)$ on $\mathbb{P}(\bigwedge^n(\mathbb{R}^{2n}))$. In practice, the numerical integration is performed on $\bigwedge^n(\mathbb{R}^{2n})$ and the formula for the Maslov angle automatically factors out the length.

In practice the unstable subspace, $\text{span}\{\zeta_1, \ldots, \zeta_n\}$, need not be computed explicitly. When the induced system on $\bigwedge^n(\mathbb{R}^{2n})$ is integrated in time, any randomly-chosen initial condition will be attracted to the most unstable direction. Hence, the Floquet multipliers or their eigenvectors do not need to be computed explicitly. It is sufficient to know that the linear system is hyperbolic. This strategy is equivalent to the *power method* for computing the eigenvalue of largest modulus of a matrix [11].

The rate of convergence of this version of the power method depends on the distance between the largest (in modulus) Floquet multiplier of $\mathbf{M}^{(n)}$ and the next largest Floquet multiplier. Explicitly, let $\mu_1, \mu_2, \ldots, \mu_n$ be the unstable Floquet multipliers with multiplicities sorted so that $1 < |\mu_1| \leq |\mu_2| \leq \ldots \leq |\mu_n|$. Then the stable Floquet multipliers are $\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_n}$. With these conditions, the two eigenvalues of largest modulus of $\mathbf{M}^{(n)}$ are $\mu_1\mu_2 \cdots \mu_n$ and $\mu_1^{-1}\mu_2 \cdots \mu_n$. The ratio of the second largest to the largest (in modulus) is

$$\frac{\mu_1^{-1}\mu_2 \cdots \mu_n}{\mu_1\mu_2 \cdots \mu_n} = \frac{1}{\mu_1^2} ,$$

and by construction this ratio has modulus strictly less than one. However the size of the ratio depends on the distance between μ_1 and the unit circle.

The Floquet multiplier of largest modulus $\mu_1\mu_2 \ldots \mu_n$ is a simple and therefore real eigenvalue. Denote the right eigenvector by $\boldsymbol{\xi} \in \bigwedge^n(\mathbb{R}^{2n})$ and let $\boldsymbol{\eta}$ represent the left eigenvector, normalized so that $\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle = 1$.

The solution of the initial-value problem for the induced system

$$U_t = \mathbf{A}^{(n)}(t)U , \quad U \in \bigwedge\nolimits^n(\mathbb{R}^{2n}) ,$$

with a random initial condition $U(t_0) = U_0$ is of the form

$$U(t) = \mathbf{\Phi}^{(n)}(t, t_0)U_0 .$$

But $\mathbf{\Phi}(t_0 + T, t_0) = \mathbf{M}^{(n)}$ and so

$$U(t_0 + kT) = \left(\mathbf{M}^{(n)}\right)^k U_0 , \quad k = 0, 1, \ldots .$$

The effect of $\left(\mathbf{M}^{(n)}\right)^k$ on the randomly chosen initial condition can be computed by the power method.

Let $U_0 := \boldsymbol{\zeta}_0$ be any randomly chosen vector such that $\langle \boldsymbol{\eta}, \boldsymbol{\zeta}_0 \rangle \neq 0$. Generically, almost every starting vector will satisfy this condition. Define the sequence $\{\boldsymbol{\zeta}_k\}$ by $\boldsymbol{\zeta}_{k+1} = \mathbf{M}^{(n)}\boldsymbol{\zeta}_k$. Then, from results on the power method, it follows that there exists a constant $C > 0$, for any $\varepsilon > 0$, such that

$$\left\| \frac{1}{\langle \boldsymbol{\eta}, \boldsymbol{\zeta}_k \rangle} \boldsymbol{\zeta}_k - \boldsymbol{\xi} \right\| \leq C(|\mu_1|^2 - \varepsilon)^{-k} , \quad k = 0, 1, 2, \ldots .$$

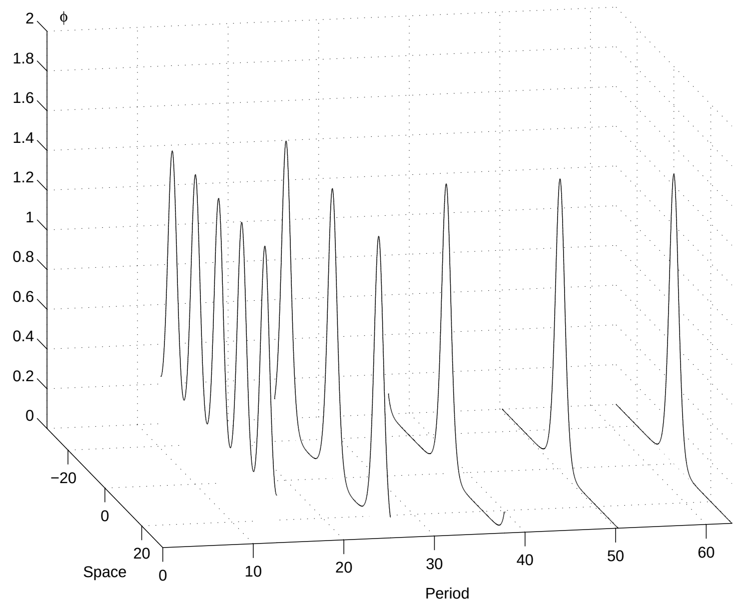

Figure 1: Periodic solutions of the steady fifth-order KdV equation for $p = 1$, $P = \frac{13}{6}$ and periods $4\pi, 8\pi, 12\pi, 16\pi, 20\pi$.

This result shows that the power method produces $\log_k(|\mu_1|^2 - \varepsilon)$ correct digits at each iteration. When $|\mu_1|$ is close to one, $\log_k(|\mu_1|^2 - \varepsilon) \sim \frac{2}{\log_e k}(|\mu_1| - 1)$. The parameter ε can be set to 0 if the eigenvalue $\mu_1^{-1}\mu_2 \cdots \mu_n$ is semi-simple. This result ensures that a random initial condition will be globally attracted to the unstable subspace, when the linear system is hyperbolic. It also demonstrates the failure of convergence in the case where the periodic orbit is not hyperbolic, e.g. when $|\mu_1| = 1$.

4 Periodic solutions of the fifth-order KdV equation

An example from the stability of waves, that illustrates the computation of the Maslov index, is *spatially*-periodic solutions of the steady fifth-order Korteweg de Vries (KdV) equation. The fifth-order KdV equation can be written in the form

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(u^{p+1}\right) + P\frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5} = 0\,, \tag{4.11}$$

where p is a natural number and P is a real parameter [6, 8]. In this paper the parameter values will be fixed at $p = 1$ and $P = \frac{13}{6}$.

Steady solutions of this equation satisfy the fourth-order ordinary differential equation

$$\phi_{xxxx} - P\phi_{xx} + \phi - \phi^{p+1} = 0. \tag{4.12}$$

In Figure 1 examples of periodic solutions of (4.12) are shown for the case of $p = 1$ and $P = \frac{13}{6}$. These solutions have been computed numerically.

The system (4.12) is Hamiltonian and the linearization about a periodic orbit leads to the self-adjoint operator in the space $L^2(\mathbb{R})$:

$$\mathscr{L}w := w_{xxxx} - P\,w_{xx} - (p+1)\phi^p\,w + w\,. \tag{4.13}$$

This operator is of interest because the linearization of (4.11) about a periodic state takes the form

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left(\mathscr{L} w \right) .$$

Properties of the spectrum of the operator $\mathscr{L}$ can be related to the spectrum of the full system (4.11) and hence can be related to stability [13]. Although it has yet to be proven, we conjecture that the Maslov index plays an important role in the stability of waves of (4.11).

Elements λ in the spectrum of $\mathscr{L}$ satisfy $\mathscr{L} w = \lambda w$ with w bounded for $x \in \mathbb{R}$ and, since $\mathscr{L}$ is self-adjoint as a mapping from $\mathscr{D}(\mathscr{L}) \to L^2(\mathbb{R})$, the spectrum is real[1]. For operators with periodic coefficients, the discrete spectrum is empty, and the spectrum consists of a sequence of bands [20].

The eigenvalue problem can be written in the form (1.1) for any $\lambda \in \mathbb{R}$,

$$\mathbf{J} \mathbf{z}_x = \mathbf{B}(x, \lambda) \mathbf{z} \,, \quad \mathbf{z} := \begin{pmatrix} w \\ w_{xx} \\ w_{xxx} - P w_x \\ w_x \end{pmatrix} ,$$

with $\mathbf{J}$ in the standard form (1.2) and

$$\mathbf{B}(x, \lambda) = \begin{pmatrix} a(x) - \lambda & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & P \end{pmatrix} , \quad a(x) = 1 - (p+1)\, \phi(x)^p \,.$$

The parameter λ plays an important role. It is a spectral parameter and it can be interpreted as a control parameter which affects whether the linear system is hyperbolic. The system is indeed hyperbolic when λ is not in the spectrum of $\mathscr{L}$.

When $\lambda = 0$ and $\phi(x)$ is periodic, this system is not hyperbolic because it is the linearization about an autonomous ODE. But when λ is perturbed away from zero the two Floquet multipliers at $+1$ move, becoming either elliptic or hyperbolic. At other values of λ, there can be bifurcations of Floquet multipliers as well; indeed, such bifurcations can be expected due to the band structure of the spectrum of $\mathscr{L}$. These issues will show up in the numerics.

The induced system on the Lagrangian Grassmannian is of the form (2.9) with

$$\mathbf{S}_1(x, \lambda) = (a(x) - \lambda - 1 + P) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

$$\mathbf{S}_2(x, \lambda) = \begin{pmatrix} 0 & 0 & a(x) - \lambda + 1 - P \\ 0 & 0 & 2 \\ -a(x) + \lambda - 1 + P & -2 & 0 \end{pmatrix} ,$$

and

$$\mathbf{N}(x, \lambda) = \begin{bmatrix} a(x) - \lambda + 1 + P & -2 & 0 \\ 0 & 0 & -a(x) + \lambda + 1 + P \end{bmatrix} .$$

[1] $\mathscr{D}(\mathscr{L})$ is the domain of the operator $\mathscr{L}$ and can be taken to be the Hilbert space $H^4(\mathbb{R})$ in this case.

Denote the period of a solution of (4.12) by L. The parameters P and p are fixed, and the parameters L and λ are varied. For each (L, λ), the Maslov index associated with the linear system is computed by integrating (2.9) over a period and computing the angle $\kappa(x, \lambda)$ using the formula (2.10) and the Maslov index is obtained using the formula (1.5).

5 Maslov index of a hyperbolic periodic orbit

When $\widehat{\mathbf{z}}(t)$ is a $T-$periodic solution of an autonomous Hamiltonian system (e.g. equation (1.6)), the linearization of the Hamiltonian system about this periodic orbit can be cast into the standard form (1.1) where $\mathbf{B}(t)$ is the Hessian of H evaluated at $\widehat{\mathbf{z}}(t)$. The linearization will have at least two Floquet multipliers at $+1$ (for definiteness, assume exactly two). Suppose that all the other Floquet multipliers are hyperbolic. The Maslov index of this orbit can be defined as follows [14, 16, 18].

The unstable manifold[2] of $\widehat{\mathbf{z}}$ is n-dimensional. Let $V(t)$ be the tangent space of the unstable manifold at $\widehat{\mathbf{z}}(t)$. $V(t)$ is spanned by $\widehat{\mathbf{z}}_t(t)$ and the unstable subspace. The unstable subspace is determined by the sum of the $(n-1)-$dimensional generalized eigenspace associated to the Floquet multipliers with modulus greater than one.

The Maslov index of $\widehat{\mathbf{z}}$ is then defined as the homotopy class of the path $t \mapsto V(t)$ in the Lagrangian Grassmannian Λ_n. If we have a matrix or n-form representation of this space, the angle κ and the homotopy class can be computed as before.

Let E_1 be the generalized eigenspace of the monodromy operator associated with the Floquet multiplier 1. E_1 is two-dimensional and contains the tangent vector $\widehat{\mathbf{z}}_t$ of the orbit.

If we ignore the fact that the linearized system is not hyperbolic and apply the same algorithm as in the previous section, there are two cases:

- $\mathbf{\Phi}(t_0+T, t_0)\big|_{E_1}$ is the identity and then taking a random starting point will not lead to convergence to $V(t)$ in general.
- $\mathbf{\Phi}(t_0+T, t_0)\big|_{E_1}$ is not the identity and then it will generally converge to $V(t)$ with the difference between the $V(t)$ and the k^{th} approximation behaving like $\frac{1}{1+k}$.

Hence, if there is convergence, it is very slow. Instead of using an n-dimensional random Lagrangian space for initial data, we can take advantage of the fact that we generally know what $\widehat{\mathbf{z}}_t$ is and:

- choose a n-dimensional Lagrangian space which already contains the $\widehat{\mathbf{z}}_t$ vector,
- choose an $(n-1)$-dimensional subspace.

These two ideas have the same rate of convergence, namely $C\frac{1}{(\mu_2-\varepsilon)^r}$ at the r^{th} iteration. For the first one, the setup is nearly identical to what has been done in the previous

[2]The unstable manifold of a periodic orbit $\widehat{\mathbf{z}}$ is defined as the union of orbits whose $\alpha-$limit set is $\widehat{\mathbf{z}}$. This should not be confused with the unstable manifold of $\widehat{\mathbf{z}}(0)$ with respect to the diffeomorphism ϕ_T which maps $u(0)$ to $u(T)$ for any solution u of the autonomous system. Of course the first includes the second but also $\widehat{\mathbf{z}}$.

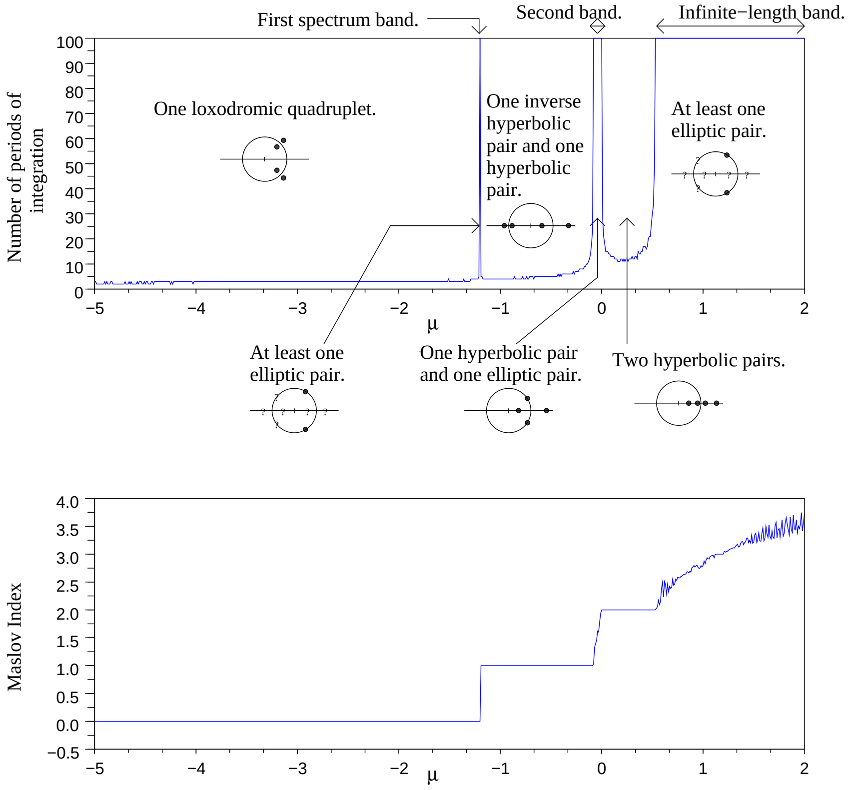

Figure 2: Maslov index of the 4π-periodic solution as a function of λ.

sections: we just pick $\widehat{\mathbf{z}}_t$ and $n-1$ other random vectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}$ and compute $\widehat{\mathbf{z}}_t(t_0) \wedge \boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_{n-1}$ for initial data.

For the second approach, we integrate until we get the unstable space of $\mathbf{\Phi}(t+T, t)$. Knowing $\widehat{\mathbf{z}}(t)$, it is then possible to compute the angle of $V(t)$. Moreover, the dimension of the space to integrate is smaller.

Another approach is to add an external parameter that perturbs the $+1$ Floquet multipliers off the unit circle. In the application to stability of waves in §4 and §6 the spectral parameter λ plays precisely this role (see Figures 2 and 6).

6 Numerical results – fifth-order KdV

In the following figures, the results of our calculations of the Maslov index are shown, for periodic solutions of (4.12) with periods $L = 4\pi, 8\pi, 16\pi, 20\pi$. For each of the four cases, two figures are shown. The first figure shows the number of iterations for the Maslov index to converge (an iteration is an integration from $x = x_0 + (r-1)L$ to $x = x_0 + rL$ for some integer r). When the system is hyperbolic, the Maslov index converges fast, and in the spectral bands the algorithm generally fails to converge. However, as shown in the second plot for each case, the Maslov index at bifurcation points is easily determined by computing in a left or right neighborhood of the bifurcation points. The Maslov index shows expected behavior for values of λ less than ≈ 1.

The static in the region λ greater than ≈ 1 is due to the appearance of spectral bands. When entering a band, as λ is varied, at least two Floquet multipliers (one inside the circle

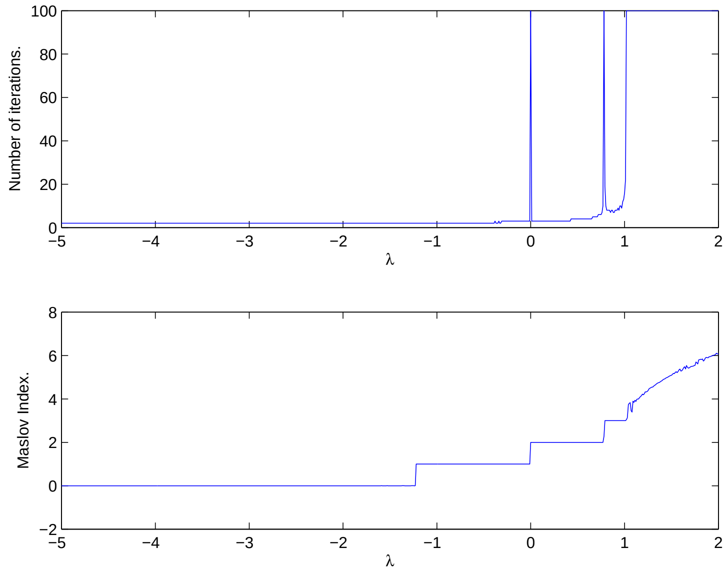

Figure 3: Maslov index of the 8π-periodic solution as a function of λ.

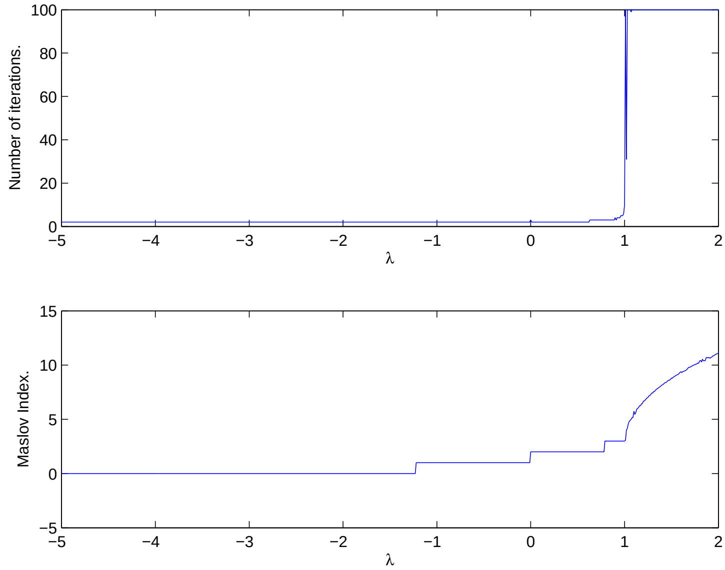

Figure 4: Maslov index of the 16π-periodic solution as a function of λ.

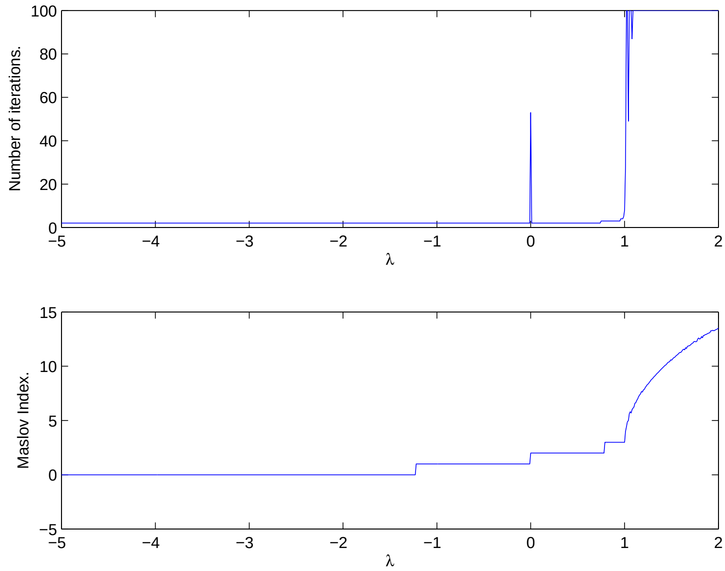

Figure 5: Maslov index of the 20π-periodic solution as a function of λ.

and the other outside) collide on the unit circle. Then they move along the unit circle. When leaving the band, Floquet multipliers collide again and leave the circle. When λ is in a band, there are Floquet multipliers on the unit circle and hyperbolicity is lost. Hence when λ is in a band the algorithm should not (and does not) in general converge.

The bands can be observed in the numerical results: the non-convergence can be used as an indication of the presence of spectrum. If the number of iterations necessary for convergence exceeded 100, it was considered an indication that the value λ was in a spectrum band or close to it. Another indication of a band, where some Floquet multipliers are on the unit circle, is the change in the value of the Maslov index: between two real where the Maslov index is different, there is at least a band.

A lower bound for all the bands can be computed which is valid for all finite wavelengths. If λ is in a band, then there exists w and γ such that $\mathscr{L}w = \lambda w$, $\forall x \quad w(x+L) = \gamma w(x)$ and $|\gamma| = 1$. Multiply $\mathscr{L}w - \lambda w$ by $\overline{w}(x)$ and integrate over a period:

$$\int_0^L |w_{xx}|^2 \,\mathrm{d}x + P\int_0^L |w_x|^2 \,\mathrm{d}x - 2\int_0^L |\phi(x)|^2|w|^2 \,\mathrm{d}x + \int_0^L |w|^2 \,\mathrm{d}x = \lambda \int_0^L |w|^2 \,\mathrm{d}x\,.$$

Now use the fact that $P > 0$ and $|\phi(x)| \leq \phi^{\max}$ for all $x \in \mathbb{R}$ to obtain

$$\lambda \geq 1 - 2|\phi^{\max}|^2\,.$$

With $|\phi^{\max}| \approx 1.4$ (from Figure (1)), we obtain a lower bound of approximately -3. This is consistent with the numerics which indicate that the lowest band is at approximately -1.25.

When the period of the periodic state $\phi(x)$ tends to ∞, the width of the bands contracts, and in the limit the continuous spectrum is limited to the one band $\lambda \in [1, \infty)$, of infinite length, and the band near zero contracts to the point $\lambda = 0$.

Consider now the case when λ is near zero. In this case the system is hyperbolic on only one side of $\lambda = 0$. When $\lambda = 0$ there are two Floquet multipliers at $+1$ and two

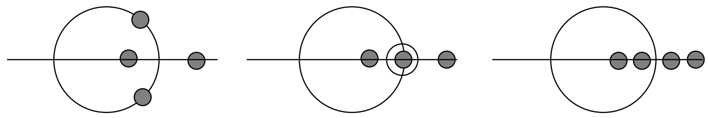

Figure 6: Position of the Floquet multipliers when going from $\lambda < 0$ to $\lambda > 0$.

hyperbolic Floquet multipliers as shown schematically in the middle figure in Figure 6. It is clear from Figure 2 that there is a band for $\lambda = 0^-$ and for $\lambda = 0^+$ the system is hyperbolic, as shown schematically in Figure 6. Our theory applies for $\lambda = 0^+$ which gives a Maslov index of 2. Therefore we *define* the Maslov index at $\lambda = 0$ to be the Maslov index in the limit $\lambda \to 0^+$.

When the system is non-hyperbolic, there are alternative definitions of the Maslov index which include corrections for the cases of elliptic and inverse hyperbolic Floquet multipliers [18, 15].

7 Concluding remarks

The computation of the Maslov index for a linear Hamiltonian system by working on the exterior algebra can be generalized to any dimension. Given a linear Hamiltonian system on $\mathbb{R}^{2n}$ of the form (1.1), let $\mathbf{A} := \mathbf{J}^{-1}\mathbf{B}(t)$. Then the induced system on $\bigwedge^n(\mathbb{R}^{2n})$ is of the form $U_t = \mathbf{A}^{(n)}U$ and with respect to the standard basis,

$$(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k})_{1 \le i_1 < i_2 < \cdots < i_k \le n}\,, \quad \text{with lexicographical ordering}\,,$$

$$(\mathbf{A}^{(n)})_{\substack{1 \le i_1 < \ldots < i_n \le n,\\ 1 \le j_1 < \ldots < j_n \le n}} = \begin{cases} 0 & \text{if } \mathrm{Card}(\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_n\}) > n+1 \\ (-1)^{r+s} a_{i_r, j_s} & \text{if } \{i_r, j_s\} = \{i_1, \ldots, i_n\} \Delta \{j_1, \ldots, j_n\} \\ \sum_{r=1}^{n} a_{i_r, i_r} & \text{if } \{i_1, \ldots, i_n\} = \{j_1, \ldots, j_n\} \end{cases}$$

with $V \Delta W = (V \cup W) - (V \cap W)$.

Suppose the basic state is hyperbolic, and let $\boldsymbol{\xi}$ represent the unstable subspace. Then the Maslov angle κ is defined by

$$\mathrm{e}^{-\mathrm{i}\frac{\kappa}{2}} = \sum_{\substack{(\{i_1 \ldots i_n\}, r) \\ i_1 < \ldots < i_n \\ \{i_1, \ldots, i_r, i_{r+1} - n, \ldots, i_n - n\} = \{1, \ldots, n\}}} \mathrm{i}^{n-r} (-1)^{\sum_{j=1}^{r} i_j - j} \boldsymbol{\xi}_{i_1, \ldots, i_n}\,.$$

Then $\frac{\kappa(T) - \kappa(0)}{2\pi}$ will give the Maslov Index. Details of this derivation and applications are given in [8].

However, the dimension of $\bigwedge^n(\mathbb{R}^{2n})$ is C^n_{2n} and it grows rapidly (for example $C^8_{16} = 12870$) and therefore the practical application of working on exterior algebra spaces is limited to low dimension. For higher dimension, *orthosymplectic integration* becomes appealing. In orthosymplectic integration, continuous orthogonalization is used and the symplectic structure is retained. For example, the algorithm proposed in [19] could be adapted to the computation of the Maslov index. In principle, if the periodic system is hyperbolic, then random orthosymplectic initial conditions can be used. However, special integrators, such as implicit Gauss-Legendre Runge-Kutta methods are required in order to preserve symplecticity and orthonormality to high accuracy, and the ODE is highly nonlinear. On the other hand, orthosymplectic integrators will be essential for systems of high dimension since the dimension grows only like $2n^2$ for linear systems on dimension $2n$.

A sketch of how orthosymplectic integration can be used is as follows. Let $\mathbf{\Phi}(t)$ be a path of symplectic matrices such that the n first columns of $\mathbf{\Phi}(t_0)$ span the unstable subspace and $\mathbf{J}\mathbf{\Phi}'(t) = \mathbf{B}(t)\mathbf{\Phi}(\mathbf{t})$. Decompose $\mathbf{\Phi}(t)$ following [19],

$$\mathbf{\Phi}(t) = \mathbf{Q}(t)\mathbf{X}(t)\,, \quad \text{with} \quad \mathbf{X}(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ 0 & X_{22}(t) \end{pmatrix} . \tag{7.14}$$

The path of matrices $\mathbf{Q}(t)$ is symplectic and orthogonal and $X_{11}(t)$ is an $n \times n$ upper triangular matrix and X_{22} is an $n \times n$ lower triangular matrix.

Since $\mathbf{Q}$ is both orthogonal and symplectic, it can be expressed in the form

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 & -\mathbf{Q}_2 \\ \mathbf{Q}_2 & \mathbf{Q}_1 \end{pmatrix} , \quad \text{with} \quad \mathbf{Q}_1 + \mathrm{i}\mathbf{Q}_2 \quad \text{unitary} .$$

If $k \leq n$, the k first columns of $\mathbf{Q}$ span the space spanned by the the first k columns of $\mathbf{\Phi}$ and therefore when $k = n$ the columns of $\begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{pmatrix}$ span the unstable space. Define κ by $e^{\mathrm{i}\kappa(t)} = \frac{\det(\mathbf{Q}_1(t) - \mathrm{i}\mathbf{Q}_2(t))}{\det(\mathbf{Q}_1(t) + \mathrm{i}\mathbf{Q}_2(t))}$, then the Maslov index is again $\frac{\kappa(T) - \kappa(0)}{2\pi}$. One still needs to prove that $\frac{\kappa(T) - \kappa(0)}{2\pi}$ is an integer, even though $\mathbf{\Phi}(t)$ is not necessarily periodic. Again the hypothesis of hyperbolicity is essential.

The only remaining problem is to compute $\mathbf{\Phi}(t_0)$ and, if the system is hyperbolic, one can take random orthosymplectic initial conditions at $t = t_0 - rT$ for some $r \in \mathbb{N}$ and perform orthosymplectic integration over $[t_0 - rT, t_0]$, then integrate the unstable subspace for $t = t_0$ to $t = t_0 + T$. The rate of convergence should be the same as the one for the exterior algebra method.

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