ON A CONJECTURE OF IGUSA

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Abstract. In his Tata Lecture Notes, Igusa conjectured the validity of a strong uniformity in the decay of complete exponential sums modulo powers of a prime number and determined by a homogeneous polynomial. This was proved for nondegenerate forms by Denef-Sperber and then by Cluckers for weighted homogeneous nondegenerate forms. In a recent preprint, J. Wright has proved this for degenerate binary forms. We give a different proof of Wright's result that seems to be simpler and relies upon basic estimates for exponential sums mod p as well as a type of resolution of singularities with good reduction in the sense of Denef.

Introduction. This article gives a fairly simple proof of a result first proved by Wright in the preprint [W]. This established "Igusa's conjecture" for any "binary form" $P(x_1, x_2) \in \mathbb{Z}[x_1, x_2]$, i.e. a homogeneous *or* weighted homogeneous polynomial (not a monomial) of degree at least two, as follows.

Theorem A. *Assume $P \in \mathbb{Z}[x_1, x_2]$ is a binary form in the above sense of degree $d \geq 2$. Then there exists $C > 0$, $h_P > 0$ and a finite set of primes $\mathcal{P}$ such that for all $p \notin \mathcal{P}$ and $r \geq 1$*

$$\Big|p^{-2r} \sum_{\boldsymbol{x} \in (\mathbb{Z}/p^r)^2} e^{2\pi w P(\boldsymbol{x}) i/p^r}\Big| \leq C r^{\kappa} \cdot p^{-r h_P} ,$$

where $\kappa \in \{0, 1\}$ depends upon p and $(w, p) = 1$.

(In fact, κ can be made uniform over p except for a very small set of special cases.)

The constant h_P in Theorem A is a particular example of what is sometimes referred to as the "weight" (or motivic oscillating index) of P. The explicit formula for h_P is simplest to give when P is homogeneous. This is given in §1 (see (1)).

Our proof is fairly simple and short. Its key elements are a type of resolution of singularities with good reduction in the sense of Denef [D] and one or more ways of estimating exponential sums over $\mathbb{F}_p$ via the work of Heath-Brown-Konyagin [HB-K], Cochrane [C], or Adolphson-Sperber [A-S]. In particular, our approach reduces the estimation problem to one in which the underlying polynomial is either a monomial in one or two variables, or the product of a monomial and polynomial in different variables. It is natural to expect that this method could extend to more than two variables.

It should be said from the outset that the estimate of Theorem A is, strictly speaking, not that conjectured by Igusa, as stated in the introduction to his Tata lectures book [Ig]. His conjecture concerned homogeneous polynomials for which the decay rate h_P *better than* 1 could be proved for any p and with a constant factor $C = C_p$ (i.e. the constant depends upon p). For such forms, he hoped that the constant factor could then be made *uniform* outside a finite set of primes. The principal motivation was, it seems, to find ways of proving an adelic Poisson summation formula like that which he showed for nonsingular projective hypersurfaces (over $\overline{\mathbb{Q}}$). This can be inferred from the fact that the property expressed by the conjecture is a hypothesis in the statement of a theorem in [ibid., ch. IV] which gives conditions under which an adelic Poisson summation formula is provable.

Since then, the work of Denef-Sperber [D-S] and Cluckers [C-1, C-2] has, to a certain degree, ignored the emphasis on the a priori decay rate. Instead the idea has been to find *some* intrinsic decay rate and then prove the existence of a constant that is uniform in p. In these papers, the Newton polyhedron has sufficed to characterize the decay rate intrinsically because the polynomials are assumed to be nondegenerate (with respect to the polyhedron). For degenerate polynomials, however, Wright [W] was the first to have reported significant progress by identifying the candidate for oscillation index and then proving the assertion for binary forms in the above sense. The same underlying idea of our proof applies to both the homogeneous or weighted homogeneous case. The approach we take is rather different from that of [ibid.].

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