# Journées du GDR TLAG 2021

## Weierstrass sections for some truncated parabolic subalgebras.

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# Motivation.

- $G$ is a connected reductive algebraic group with Lie algebra $\mathfrak{g}$.
  - For instance $G = GL_n(\mathbb{C})$, with Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$, which is the Lie algebra of all complex $n \times n$ matrices endowed with Lie bracket $[x, y] = xy - yx$ for $x, y \in \mathfrak{gl}_n$.

**Theorem (Chevalley)**

*The algebra $Y(\mathfrak{g}) := \mathbb{C}[\mathfrak{g}^*]^G$ of invariant polynomial functions on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is a polynomial algebra over $\mathbb{C}$ that is, this algebra is generated by a finite number of invariant functions which are homogeneous and algebraically independent.*

# Chevalley's theorem.

Chevalley's theorem follows from the following result :

- Denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and by $W$ the Weyl group of $(\mathfrak{g},\ \mathfrak{h})$.
  - For instance if $\mathfrak{g} = \mathfrak{gl}_n$, then $\mathfrak{h}$ may be taken to be the vector space generated by the diagonal matrices, and $W = \mathfrak{S}_n$ acts on $\mathfrak{h}$ on the obvious way.
- Then the restriction map $res_1$

$$\begin{array}{rccc} res_1 : & Y(\mathfrak{g}^*) = \mathbb{C}[\mathfrak{g}]^G & \xrightarrow{\sim} & \mathbb{C}[\mathfrak{h}]^W \\ & f & \longmapsto & f_{|\mathfrak{h}} \end{array}$$

  is an algebra isomorphism.
- Since $W$ is a finite subgroup of $GL(\mathfrak{h})$ generated by reflections, then $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial algebra.

# Kostant section.

## Theorem (Kostant)

*Let $\mathfrak{g}$ be a reductive Lie algebra. Then there exists an affine subspace $\mathcal{S}$ of $\mathfrak{g}^*$ such that the restriction map*

$$\begin{array}{rccc} res: & Y(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]^G & \xrightarrow{\sim} & \mathbb{C}[\mathcal{S}] \\ & f & \longmapsto & f_{|\mathcal{S}} \end{array}$$

*is an algebra isomorphism.*
*$\mathcal{S}$ is called the Kostant section or Kostant slice.*

### Definition

If $\mathfrak{g}$ is reductive, there exists a principal $\mathfrak{sl}_2$-triple $(x, h, y)$ of $\mathfrak{g}$ : that is, $x$ and $y$ are regular in $\mathfrak{g}^* \simeq \mathfrak{g}$ - the codimension of their $G$-orbit is minimal, equal to the index of $\mathfrak{g}$ - and $h$ is semisimple and such that $[h, y] = -y$, and $[x, y] = h$.

The Kostant section constructed by Kostant is

$$\mathcal{S} = y + \mathfrak{g}^x.$$

- For $\mathfrak{g} = \mathfrak{gl}_n$, the element $y$ may be taken to be equal to the matrix with entries 1 under the first diagonal and zero elsewhere. Then $x$ and $h$ can be chosen suitably (by the Jacobson-Morosov theorem).

- The Kostant section was constructed from a principal $\mathfrak{sl}_2$-triple $(x,\ h,\ y)$ : the vector space generated by this triple is isomorphic to $\mathfrak{sl}_2$, and $\mathfrak{g}$ is an $\mathfrak{sl}_2$-module.
  Then by $\mathfrak{sl}_2$-theory, one has that $\mathfrak{g} = [\mathfrak{g},\ y] \oplus \mathfrak{g}^x$.
  One remarks that the Kostant section $\mathcal{S} = y + \mathfrak{g}^x$ with $\mathfrak{g}^x$ being an $h$-stable complement of the $G$-orbit of $y$.
- The eigenvalues of $h$ on $\mathfrak{g}^x$ are all nonnegative integers $m_i$ $(1 \leq i \leq \mathrm{rk}\,\mathfrak{g} = \mathrm{index}\,\mathfrak{g})$.
  Actually they are the exponents of $\mathfrak{g}$, namely $m_i + 1 = \deg(f_i)$ where the set $\{f_i\ ;\ 1 \leq i \leq \mathrm{rk}\,\mathfrak{g}\}$ is a set of homogeneous algebraically independent generators of $Y(\mathfrak{g})$.
- $\sum_{i=1}^{\mathrm{rk}\,\mathfrak{g}} \deg(f_i) = c(\mathfrak{g}) := 1/2(\dim\mathfrak{g} + \mathrm{rk}\,\mathfrak{g})$.

# Weierstrass section and adapted pair.

## Definition (Popov)

Let $\mathfrak{a}$ be an algebraic complex finite dimensional Lie algebra (not necessarily reductive) and $A$ its adjoint group.
An affine subspace $\mathcal{S}$ of $\mathfrak{a}^*$ verifying the same property as the Kostant section is called a Weierstrass section for coadjoint action of $\mathfrak{a}$. In other words, $\mathcal{S}$ is a Weierstrass section when

$$\begin{array}{rcccc} res: & Y(\mathfrak{a}) = \mathbb{C}[\mathfrak{a}^*]^A & \xrightarrow{\sim} & \mathbb{C}[\mathcal{S}] \\ & f & \longmapsto & f_{|\mathcal{S}} \end{array}$$

is an algebra isomorphism.

## Definition (Joseph-Lamprou)

An adapted pair for $\mathfrak{a}$ is a pair $(h, y) \in \mathfrak{a} \times \mathfrak{a}^*$ such that $h$ is semisimple, $y$ is regular in $\mathfrak{a}^*$ and $(ad^* h)(y) = -y$.

# An analogue of Kostant's theorem.

**Theorem (Joseph-Shafrir)**

*Assume that*

- $Y(\mathfrak{a}) = \mathbb{C}[\mathfrak{a}^*]^A$ *is polynomial,*
- *there exists no proper semi-invariant in* $\mathbb{C}[\mathfrak{a}^*]$,
- *an adapted pair* $(h, y)$ *for* $\mathfrak{a}$ *exists,*
- *there exists a vector space* $V \subset \mathfrak{a}^*$, $ad^* h$*-stable, such that* $(ad^*\mathfrak{a})(y) \oplus V = \mathfrak{a}^*$.

**Theorem (Joseph-Shafrir)**

*Then*

- $\dim V = \mathrm{index}\,\mathfrak{a} = \min_{x \in \mathfrak{a}^*} \operatorname{codim} A.x$.
- *$y + V$ is a Weierstrass section for coadjoint action of $\mathfrak{a}$.*
- *The eigenvalues of $h$ on $V$ are all nonnegative integers $m_i$ and $m_i + 1$ is the degree of each homogeneous generator $f_i$ of $Y(\mathfrak{a})$.*
- $$\sum_{i=1}^{\mathrm{index}\,\mathfrak{a}} \deg(f_i) = c(\mathfrak{a}) := 1/2(\dim \mathfrak{a} + \mathrm{index}\,\mathfrak{a})$$

# Affine slice.

**Theorem (Joseph-FM)**

*Assume that $\mathcal{S} \subset \mathfrak{a}^*$ is a Weierstrass section for coadjoint action of $\mathfrak{a}$, and that there is no proper semi-invariant in $\mathbb{C}[\mathfrak{a}^*]$. Then $\mathcal{S}$ is an affine slice, which means that :*

- $\overline{A.\mathcal{S}} = \mathfrak{a}^*$ *and*
- *every coadjoint orbit in $\mathfrak{a}^*$ meets $\mathcal{S}$ in at most one point and transversally.*

# Our aim.

- Our aim is to construct Weierstrass sections for some particular non reductive algebraic Lie algebras $\mathfrak{a}$ (where there is no proper semi-invariant in $\mathbb{C}[\mathfrak{a}^*]$) thanks to adapted pairs for $\mathfrak{a}$.
- Then these Weierstrass sections will also be affine slices.

# Some remarks.

## Remarks

- *The existence of a Weierstrass section implies the polynomiality of $Y(\mathfrak{a}) = \mathbb{C}[\mathfrak{a}^*]^A$. But the latter is not sufficient.*
- *Take $\mathfrak{a} = \mathfrak{n}$, a maximal nilpotent subalgebra of a simple Lie algebra of type $\mathrm{C}_2$, the Lie algebra of the symplectic Lie group $Sp_2(\mathbb{C})$.*
  *Then $Y(\mathfrak{n})$ is polynomial but there exists no Weierstrass section for coadjoint action of $\mathfrak{n}$.*
  *Indeed a Weierstrass section gives a linearization of invariant generators of $Y(\mathfrak{a})$ and in case $\mathfrak{a} = \mathfrak{n}$ in type $\mathrm{C}_2$, such a linearization is not possible.*
- *An adapted pair does not necessarily exist, even when a Weierstrass section exists, as it occurs for instance for the truncated Borel subalgebra in type B.*

# Our principal result.

Let $\mathfrak{p}$ be a parabolic subalgebra of a simple Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_N$ in type $\mathrm{B_n}$, $\mathrm{C_n}$, $\mathrm{D_n}$ with a Levi factor formed (on each side of the second diagonal) by a first and a last block and eventually between them, blocks of size two : this also includes the maximal parabolic subalgebras, that is, with a Levi factor composed by two blocks on each side of the second diagonal.

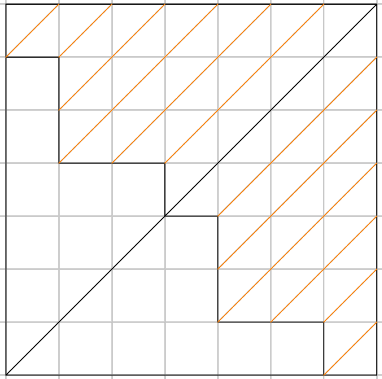
Let $\mathfrak{p}_\Lambda$ be the canonical truncation of $\mathfrak{p}$ : so that there is no proper semi-invariant in $\mathbb{C}[\mathfrak{p}_\Lambda^*]$ and the algebra of semi-invariant polynomial functions on $\mathfrak{p}^*$ is equal to $Y(\mathfrak{p}_\Lambda)$. In (most) of the above cases $\mathfrak{p}_\Lambda$ is just the derived subalgebra $\mathfrak{p}' = [\mathfrak{p}, \mathfrak{p}]$ of $\mathfrak{p}$.

**Theorem (Lamprou-FM and FM)**

*With the above hypotheses, $Y(\mathfrak{p}_\Lambda)$ is a polynomial algebra and there exists a Weierstrass section for coadjoint action of $\mathfrak{p}_\Lambda$. Moreover this Weierstrass section is given by an adapted pair of $\mathfrak{p}_\Lambda$.*

# An example of a truncated parabolic subalgebra.

*Parabolic* $\mathfrak{p}$ *in type* $B_3$ *associated with* $\{\alpha_2\}$



*Here* $Y(\mathfrak{p}')$ *is a polynomial algebra*

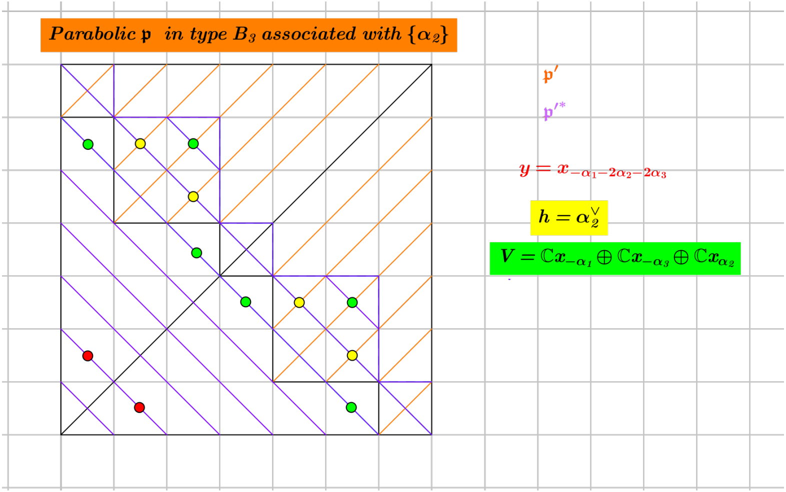
*Moreover there exists a Weierstrass section for coadjoint action of* $\mathfrak{p}'$

# Example with $\mathfrak{p}$ in type $\mathrm{B}_3$ associated with $\{\alpha_2\}$.

Here the canonical truncation of $\mathfrak{p}$ is $\mathfrak{p}_\Lambda = \mathfrak{p}'$ and take Bourbaki's labelling for the simple roots $\alpha_1,\ \alpha_2,\ \alpha_3$. Denote by $x_\alpha$ one nonzero root vector in the root space corresponding to the root $\alpha$ and denote by $\alpha^\vee$ the coroot of $\alpha$. Note that $\mathfrak{p}'^*$ is isomorphic to the opposite parabolic subalgebra of $\mathfrak{p}'$.

- There exists an adapted pair $(h,\ y)$ for $\mathfrak{p}'$ (it is not unique in general). One may take $y = x_{-\alpha_1-2\alpha_2-2\alpha_3}$ and $h = \alpha_2^\vee$.
- Take $V = \mathbb{C}x_{-\alpha_1} \oplus \mathbb{C}x_{-\alpha_3} \oplus \mathbb{C}x_{\alpha_2}$ and $\mathcal{S} = y + V \subset \mathfrak{p}'^*$. One has that
  - $(ad^*\mathfrak{p}')(y) \oplus V = \mathfrak{p}'^*$
  - $\dim V = \mathrm{index}\,\mathfrak{p}' = \mathrm{degtr}_{\mathbb{C}}(Y(\mathfrak{p}'))$.

# More on the figure.



# Example continue.

- $$\begin{array}{cccc} res: & Y(\mathfrak{p}') & \stackrel{\sim}{\longrightarrow} & \mathbb{C}[\mathcal{S}] \\ & f & \longmapsto & f_{|\mathcal{S}} \end{array}$$
  is an algebra isomorphism that is, $\mathcal{S}$ is a Weierstrass section for coadjoint action of $\mathfrak{p}'$ and also an affine slice.
- The eigenvalues of $h$ on $V$ are $1,\ 1,\ 2$, then the degrees of homogeneous algebraically independent generators $f_1,\ f_2,\ f_3$ of $Y(\mathfrak{p}')$ are : $2,\ 2,\ 3$.
- $c(\mathfrak{p}') = 1/2(\dim \mathfrak{p}' + \mathrm{index}\, \mathfrak{p}') = 1/2(11+3) = 7 = \sum_{i=1}^{3} \deg(f_i)$.

# Perspectives.

- Let $\mathfrak{p}$ be a parabolic subalgebra of a simple Lie algebra : $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ where $\mathfrak{r}$ is reductive and $\mathfrak{m}$ is the nilpotent radical of $\mathfrak{p}$.
- Consider the contraction $\tilde{\mathfrak{p}} = \mathfrak{r} \ltimes (\mathfrak{m})^a$ where $(\mathfrak{m})^a$ becomes an abelian ideal of $\tilde{\mathfrak{p}}$. Such a contraction was considered by Panyushev and Yakimova but also by Feigin for other examples of Lie algebras : they are also called Inonü-Wigner contractions.
- One would like to use the Weierstrass sections obtained for coadjoint action of $\mathfrak{p}_\Lambda = \mathfrak{p}'$ when $\mathfrak{p}$ is maximal to hope to obtain a Weierstrass section for coadjoint action of $(\tilde{\mathfrak{p}})_\Lambda = \tilde{\mathfrak{p}}' = \mathfrak{r}' \ltimes (\mathfrak{m})^a$.
- The Lie algebra $\mathfrak{s} = \mathfrak{r}'$ is semisimple (in general not simple) and Panyushev and Yakimova have studied the polynomiality of the algebra of invariants for the semi-direct product $\mathfrak{s} \ltimes V^a$, but when $\mathfrak{s}$ is simple (and in type A, only for one type of representation $V$).