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A CRITERION FOR GREENBERG'S CONJECTURE

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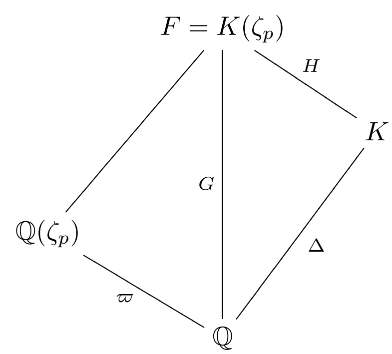
Abstract. We give a criterion for the vanishing of the Iwasawa λ-invariants of totally real number fields K based on the class number of $K(\zeta_p)$ by evaluating the p-adic L-functions at $s = -1$.

1. Introduction

Let K be a real abelian number field and let p be an odd prime. Set $F = K(\zeta_p)$ where ζ_p is a primitive p-th root of unity and $H = \mathrm{Gal}(F/K)$. Set, moreover, $G = \mathrm{Gal}(F/\mathbb{Q})$ and $\varpi = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. So the diagram of our extensions is as follows:



Let $\tilde{\omega} : H \to \mathbb{Z}_p^\times$ and $\omega : \varpi \to \mathbb{Z}_p^\times$ be the Teichmüller characters of K and $\mathbb{Q}$, respectively. We give (Theorem 2.3) a criterion under which a set of odd Iwasawa invariants associated with F vanishes: by means of a Spiegelungsatz, these odd invariants make their even mirrors vanish too. In the case $p = 3$ (Corollary 2.5) or $p = 5$ and $[K : \mathbb{Q}] = 2$ (Theorem 2.7) this allows us to verify a conjecture of Greenberg for the fields satisfying our criterion.

2. Main result

Proposition 2.1. *The following equality holds:*

$$rk_p\big(\mathrm{K}_2(\mathcal{O}_K)\big) = rk_p\big((Cl'_F)_{\omega^{-1}}\big) + |S|,$$

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where $\mathrm{K}_2(\mathcal{O}_K)$ *is the tame kernel of* K*-theory,* Cl'_F *is the class group of the ring* $\mathcal{O}_F[1/p]$ *(and we take its* $\tilde{\omega}^{-1}$*-component for the action of* H*) and* S *is the set of* p*-adic primes of* K *which split completely in* F*.*

Proof. This result dates back to Tate: for an explicit reference, see [Gra], Theorem 7.7.3.1. □

Proposition 2.2. *Suppose that* $\mathbb{Q}(\zeta_p)$ *is linearly disjoint from* K *over* $\mathbb{Q}$*. Then the following equalities hold:*

$$v_p\big(|\mathrm{K}_2(\mathcal{O}_K)|\big) = v_p\big(\zeta_K(-1)\big) \quad \text{if } p \geq 5 \ ,$$
$$v_3\big(|\mathrm{K}_2(\mathcal{O}_K)|\big) = v_3\big(\zeta_K(-1)\big) + 1,$$

where v_p *denotes the standard* p*-adic valuation and* ζ_K *is the Dedekind zeta function for* K*.*

Proof. The Birch-Tate conjecture, which has been proved by Mazur, Wiles and by Greither (since it is a consequence of the Main Conjecture in Iwasawa theory), tells us that

$$\frac{|K_2(\mathcal{O}_K)|}{w_2} = \zeta_K(-1),$$

where

$$w_2 = \max\{n \in \mathbb{N} \,|\, \text{the exponent of } \mathrm{Gal}(K(\zeta_n)/K) \text{ is } 2\}.$$

By our hypothesis, $\mathbb{Q}(\zeta_p)$ is linearly disjoint from K over $\mathbb{Q}$. Hence F/K is Galois with cyclic Galois group of order $p-1$. If $p=3$, then for the same argument, $3|w_2$ but $9 \nmid w_2$ since $K(\zeta_9)/K$ has degree 6. Taking p-adic valuation we get the claim. □

Theorem 2.3. *Let* $p \geq 5$*. Suppose that the following hold:*

- K *and* $\mathbb{Q}(\zeta_p)$ *are linearly disjoint over* $\mathbb{Q}$*;*
- *the set* S *of Proposition* 2.1 *is empty;*
- *the Main Conjecture of Iwasawa theory holds for* F*.*

Then, if p *does not divide the order of* $Cl_F(\tilde{\omega}^{-1})$*,* $\lambda_{\chi\omega^2}(F) = 0$ *for all characters* χ *of* Δ*.*

Proof. First of all, we should just prove the theorem for nontrivial characters of Δ, since $\lambda_{\omega^2} = 0$ as it corresponds to the ω^2-part of the cyclotomic extension of $\mathbb{Q}(\zeta_p)$, which is always trivial: indeed, $B_{1/2} = -1/2$, and then Herbrand's theorem and Leopoldt's Spiegelungsatz ([Was], Theorems 6.7 and 10.9) give $\lambda_{\omega^2} = 0$.

By hypothesis, the set S of Proposition (2.1) is empty. Therefore $rk_p(\mathrm{K}_2(\mathcal{O}_K)) = 0$ and Proposition (2.2) (which we can apply because K verifies its hypothesis) together with $p \geq 5$ tells us that $v_p(\zeta_K(-1)) = 0$. Since we can factor

$$\zeta_K(s) = \prod_{(\chi \in \hat{\Delta})} L(s,\chi) = \zeta_{\mathbb{Q}}(s) \prod_{\chi \neq 1} L(s,\chi),$$

we find that

$$v_p\big(\zeta_K(-1)\big) = \sum_{\chi \neq 1} v_p\big(L(-1,\chi)\big) = 0 \ . \tag{2.1}$$

The interpolation formula for the p-adic L-function (see [Was], Chapter 5) tells us that

$$L_p(-1,\chi) = \big(1 - \chi\omega^{-2}(p)p\big)L(-1,\chi\omega^{-2}); \tag{2.2}$$

now we invoke the Main Conjecture as stated in ([Gre], page 452) to relate these L-functions with the characteristic polynomials of some submodules of the Iwasawa module $X_\infty(F)$. Observe that the hypothesis of linear disjointness tells us that $\hat{G} \cong \hat{\Delta} \times \hat{\varpi}$, so we can split

$$X_\infty(F) \cong \bigoplus_{\chi\in\Delta} \bigoplus_{i=1}^{p-1} X_\infty(F)(\chi\omega^i),$$

where G acts on $X_\infty(F)(\chi\omega^i)$ as $g{\cdot}x = (\chi\omega^i)(g)x$ for all $g \in G$ and $x \in X$. Then the Main Conjecture for F allows us to write $L_p(-1, \chi\omega^i) = f(-p/(1+p), \chi^{-1}\omega^{1-i})$ for all even $2 \le i \le p-1$, where $f(T, \chi^{-1}\omega^{1-i}) \in \mathbb{Z}_p[T]$ is the characteristic polynomial of $X_\infty(F)(\chi^{-1}\omega^{1-i})$: thus $L_p(-1, \chi\omega^i)$ is $\mathbb{Z}_p$-integral. Applying this for $i = 2$ and plugging it in (2.2) we find $v_p\big(L(-1, \chi)\big) \ge 0$ for all χ, and thanks to (2.1) we indeed find $v_p\big(L(-1, \chi)\big) = 0$ for all $\chi \in \hat{\Delta}$, so

$$v_p\big(L_p(-1, \chi\omega^2)\big) = 0 \qquad \forall\, \chi \in \hat{\Delta}\ .$$

If we now apply again the Main Conjecture, we find that this corresponds to

$$v_p\Big(f\big(\frac{1}{1+p} - 1, \chi^{-1}\omega^{-1}\big)\Big) = v_p\Big(f\big(\frac{-p}{1+p}, \chi^{-1}\omega^{-1}\big)\Big) = 0 \qquad \forall\, \chi \in \hat{\Delta}\ .$$

Since $f(T, \chi^{-1}\omega^{-1}) \in \mathbb{Z}_p[T]$ is distinguished (see [Was], Chapter 7) this is possible if and only if $\deg_T\big(f(T, \chi^{-1}\omega^{-1})\big) = 0$; but this is precisely the Iwasawa invariant $\lambda_{\chi^{-1}\omega^{-1}}$, so we have

$$\lambda_{\chi^{-1}\omega^{-1}} = 0 \qquad \forall\, \chi \in \hat{\Delta}\ .$$

Since the inequality $\lambda_{\chi^{-1}\omega^{-1}} \ge \lambda_{\chi\omega^2}$ is classical and well known (see, for instance, [BN], Section 4), we achieve the proof. □

Remark 2.4. We should ask that the Main Conjecture holds for K to apply it in the form of [Gre]. For this, it is enough that there exists a field E that is unramified at p and such that $F = E(\zeta_p)$, as is often the case in the applications. Moreover, we remark that the hypotheses of the theorem are trivially fulfilled if p is unramified in $K/\mathbb{Q}$.

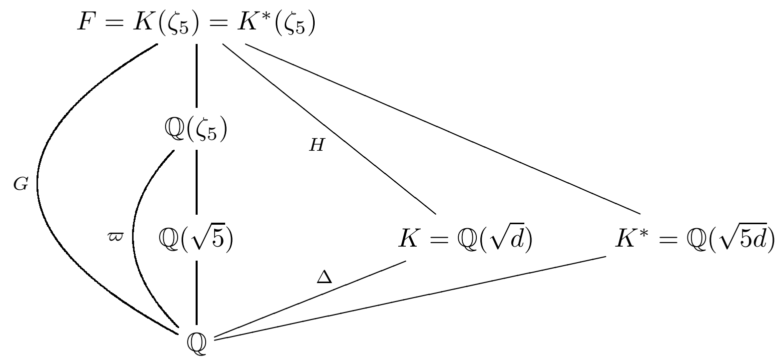
Corollary 2.5. *Assume $p = 3$. If 3 does not divide the order of $Cl_F(\tilde{\omega}^{-1})$ and it is unramified in K, then $\lambda(K) = \lambda(F) = 0$.*

Proof. First of all, the theorem applies for $p = 3$ also, since we still have (2.1) thanks to $\zeta(-1) = -1/12$; moreover, K is clearly disjoint from $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, as it is unramified, and F/K is ramified, so $S = \emptyset$. But in this case we have $\omega^2 = 1$, so the statement of the theorem is that all Iwasawa invariants λ_χ vanish for $\chi \in \hat{\Delta}$ and their sum is precisely $\lambda(K)$. Concerning $\lambda(F)$, in the proof of the theorem we first prove that all $\lambda_{\chi\omega}$ vanish, and deduce from it the vanishing of their "mirror" parts. □

Remark 2.6. In the case that $K = \mathbb{Q}(\sqrt{d})$ is real quadratic, this is a classical result of Scholtz (although it is expressed in term of Iwasawa invariants); see [Was], Theorem 10.10.

Theorem 2.7. *Let K be a real quadratic field and suppose that $5 \nmid |Cl_F|$. Then $\lambda(K) = 0$.*

Proof. Write $K = \mathbb{Q}(\sqrt{d})$ and let χ be its nontrivial character: the result being well known if $d = 5$ we assume throughout that $d \neq 5$. Then we should consider two cases, namely $5 \mid d$ and $5 \nmid d$. We have the following diagram of fields (we don't draw the whole of it):



Suppose first of all that $5 \mid d$ or that 5 is inert in $K/\mathbb{Q}$. Since $5 \nmid [F : K]$, our hypothesis implies that $5 \nmid |Cl_K|$ (see [Was], Lemma 16.15). But then we would trivially have $\lambda(K) = 0$ as an easy application of Nakayama's Lemma (see [Was], Proposition 13.22). We can thus suppose that 5 splits in $K/\mathbb{Q}$. We then apply Theorem 2.3 to K^* instead of K: since $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\zeta_5)$, our field is linearly disjoint over $\mathbb{Q}$ from $\mathbb{Q}(\zeta_5)$ and $S = \emptyset$ thanks to degree computations. Moreover the Main Conjecture holds for F since $F = K(\zeta_5)$ and K is totally real and unramified at 5. We find that $\lambda_{\omega^2\chi^*} = 0$ where χ^* is the nontrivial character of K^*. But clearly $\chi^* = \chi\omega^2$, so $\lambda_\chi = 0$. Since the Iwasawa invariant associated with the trivial character is $\lambda(\mathbb{Q}) = 0$, we have $\lambda(K) = \lambda(\mathbb{Q}) + \lambda_\chi = 0$. $\square$

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