Hankel-type determinants and Drinfeld quasi-modular forms

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In the memory of David Hayes

Abstract. In this paper we introduce a class of determinants “of Hankel type”. We use them to compute certain remarkable families of Drinfeld quasi-modular forms.

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1 Introduction

It is well known that the $\mathbb{Z}$-algebra $M_{\mathbb{Z}}$ generated by classical modular forms whose $q$-expansion has coefficients in $\mathbb{Z}$ is finitely generated, being the polynomial algebra $\mathbb{Z}[E_4, E_6, \Delta]$ where $E_4, E_6$ are the normalised \textsuperscript{(1)} Eisenstein series of weights 4, 6 respectively, and where $\Delta = (E_4^3 - E_6^2)/1728$ is the unique normalised cusp form of weight 12.

Let $\widetilde{M}_Q$ be the $\mathbb{Q}$-algebra of classical quasi-modular forms, as defined by Kaneko and Zagier in \cite{7}, whose $q$-expansions have coefficients in $\mathbb{Q}$. It is easy to show that $\widetilde{M}_Q = \mathbb{Q}[E_2, E_4, E_6]$, where $E_2$ is the (non-modular) normalised Eisenstein series of weight 2. We may then formulate the following:

\textbf{Problem 1.} \textit{Compute a minimal set of generators for $\widetilde{M}_{\mathbb{Z}} = \widetilde{M}_Q \cap \mathbb{Z}[[q]]$, the $\mathbb{Z}$-algebra generated by quasi-modular forms of $\widetilde{M}_Q$ whose $q$-expansions have coefficients in $\mathbb{Z}$.}

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\textsuperscript{1}A formal power series $\sum_{i \geq i_0} c_i q^i$ (or $\sum_{i \geq i_0} c_i u^i$) is said to be normalised if $c_{i_0} = 1$. A modular form is normalised, by definition, if its $q$-expansion is normalised. A similar definition will be used for quasi-modular forms and for Drinfeld quasi-modular forms.

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This problem is likely to be a difficult one. The examination of the \( q \)-expansion of the quasi-modular forms \( DE_w \) with \( E_w \) normalised Eisenstein series of weight \( w \), \( D = q dq/dq \) and Clausen-von Staudt Theorem, indicate that the algebra \( \tilde{M}_2 \) is likely to be not finitely generated, in contrast with the structure of \( M_2 \). But then, what does a minimal set of generators of \( \tilde{M}_2 \) look like?

In [8], Kaneko and Koike introduced a notion of extremal quasi-modular form (\(^2\)). An extremal quasi-modular form of weight \( w \) and depth \( \leq l \) is a non-vanishing polynomial in \( E_2, E_3, E_6 \) which is isobaric of degree \( w \), whose degree in \( E_2 \) is not bigger than \( l \), and such that the order of vanishing at \( q = 0 \) of its \( q \)-expansion is maximal. If \( w \geq 0 \) is even and \( l \geq 0 \), such a form exists and is multiple to a unique normalised form in \( M_2 \) denoted by \( f_{l,w} \).

Kaneko and Koike, in [8] Conjecture 2), made a prediction on the size of the denominators of such coefficients which resembles in some way to a generalisation of Clausen-von Staudt Theorem: if Conjecture 2 of loc. cit. holds, then \( f_{l,w} \in \mathbb{Z}_p[[q]] \) for every prime number \( p \) such that \( p \geq w \), provided that \( l \leq 4 \). In addition, the following question can be addressed.

**Question.** Let \( l \) be a non-negative integer, and denote by \( \mathcal{E}_l \) the set of \( w \)'s such that \( f_{l,w} \) exists, and belongs to \( \mathbb{Z}[[q]] \). For which \( l \)'s is \( \mathcal{E}_l \) infinite?

Although very few of the \( f_{l,w} \)'s are known to have \( q \)-expansion defined over \( \mathbb{Z} \) (\(^3\)), the feeling that we have, after several extensive numerical computations, is that \( \mathcal{E}_l \) is infinite for \( 0 \leq l \leq 4 \) and finite for \( l > 4 \). In particular, the existence of infinitely many forms \( f_{l,w} \) with \( q \)-expansion defined over \( \mathbb{Z} \) would suggest to use them to construct a set of generators for \( \tilde{M}_2 \) but we refrain from making any kind of written prediction in this direction because this hypothesis is, so far, largely conjectural.

In this paper, we want to discuss similar problems, arising in the theory of Drinfeld quasi-modular forms, where we have a slightly better understanding of what is going on. Let \( q = p^e \) be a power of a prime number \( p \) with \( e > 0 \) an integer, let \( \mathbb{F}_q \) be the finite field with \( q \) elements. Let us consider, for an indeterminate \( \theta \), the polynomial ring \( A = \mathbb{F}_q[\theta] \) and its fraction field \( K = \mathbb{F}_q(\theta) \).

Let \( K_\infty \) be the completion of \( K \) for the \( \theta^{-1} \)-adic valuation and let us embed an algebraic closure of \( K_\infty \) in its completion \( C_\infty \) for the unique extension of that valuation. Following Gekeler in [2], we denote by \( \Omega \) the set \( C_\infty \setminus K_\infty \), which has a structure of a rigid analytic space over which the group \( \Gamma = \text{GL}_2(A) \) acts discontinuously by homographies, and with the usual local parameter at infinity \( u \) (denoted by \( t \) in [2]). These facts lead quite naturally to the notion of Drinfeld quasi-modular forms, rather parallel to that of classical quasi-modular forms for \( \text{SL}_2(\mathbb{Z}) \), which are studied in [2], and to which we refer for the required background.

Following [2], we have three remarkable formal series \( E, g, h \in A[[u]] \) algebraically independent over \( K(u) \), representing respectively: the \( u \)-expansion of a Drinfeld quasi-modular form of weight 2, type 1 and depth 1 (the false Eisenstein series of weight 2 of Gekeler [2]), the \( u \)-expansion of an Eisenstein series of weight \( q-1 \) and type 0, and the \( u \)-expansion of a Poincaré series of weight \( q+1 \) and type 1. The first terms of these formal series are as follows, where \( [i] = \theta^i - \theta \) (see [2] Lemma 4.2)): \(^4\)

\[
E = u + uq^2 - 2q + 2 + \cdots \in u A[[u^{q-1}]] \\
g = 1 - [1]u^{q-1} - [1]uq^{2-2q^2+2q-1} + \cdots \in A[[u^{q-1}]] \\
h = -u - uq^2 - 2q^2 + 2 + \cdots \in u A[[u^{q-1}]].
\]

\(^4\)Notice that in fact, the definition of extremality of Kaneko and Koike slightly differs from ours.

\(^3\)For example, it is not known whether \( f_{1,14} \in \mathbb{Z}[[q]] \) but this looks like true from numerical evidence.
Let $M_{w,m}$ be the $K$-vector space of Drinfeld modular forms of weight $w$, type $m$, whose $u$-expansions are defined over $K$, which also is the space of isobaric polynomials (for weights and types) in $g$ and $h$ with coefficients in $K$ (4). The $K$-vector space of Drinfeld quasi-modular forms of weight $w$, type $m$ and depth $\ell$, defined over $K$ is, by definition, the space

$\tilde{M}_{w,m}^{\leq \ell} = M_{w,m} \oplus M_{w-2,m-1}E \oplus \cdots \oplus M_{w-2\ell,m-\ell}E^\ell$. 

All these spaces are finite dimensional subspaces of $K[[u]]$ and we may form the $C^\infty$-algebra of Drinfeld quasi-modular forms

$\tilde{M}_K = \bigoplus_{w,m} \bigcup_{\ell} \tilde{M}_{w,m}^{\leq \ell}$.

In analogy with the Problem 1, we have:

**Problem 2.** Compute a minimal set of generators for $\tilde{M}_A$, the $A$-algebra generated by quasi-modular forms of $\tilde{M}_K$ whose $u$-expansions have coefficients in $A$.

We say that an element $f$ of $\tilde{M}_{w,m}^{\leq \ell} \setminus \{0\}$ is an extremal Drinfeld quasi-modular form if $\text{ord}_u = 0$ is maximal among the orders at $u = 0$ of non-zero elements of that vector space. If there exists an extremal Drinfeld quasi-modular form of $\tilde{M}_{w,m}^{\leq \ell}$ (5), we denote by $f_{i,w,m}$ the unique normalised such form.

To present our main result, we need to define a double sequence of quasi-modular forms

$$(E_{j,k})_{j \in \mathbb{Z}, k \geq 1}.$$ 

Let us write $[j] = \theta q^j - \theta$ for $j \in \mathbb{Z}$. The sub-sequence $(E_{j,1})_{j \in \mathbb{Z}}$ is defined inductively in the following way. We set $E_{0,1} = E$, $E_{1,1} = -\frac{E_{q+1}}{q}$ and then, for $j \geq 0$, by

$$E_{j+2,1} = -\frac{1}{[j+2]}(\Delta^{q^j} E_{j,1} + g^{q^{j+1}} E_{j+1,1}),$$

as for $j \leq 1$, by

$$E_{j-2,1} = -\frac{1}{\Delta^{q^{j-2}}}( [j] E_{j,1} + g^{q^{j-1}} E_{j-1,1}).$$

For example, we have the following particular cases:

$$E_{-1,1}^q = -h,$$

$$E_{-2,1}^q = -h g^q,$$

$$E_{-3,1}^q = -h (g^{q+1} - [1]^q h^{q-1})^q,$$

and in general, for all $j \leq -1$, it is possible to check that $E_{-j,1}^q$ is a Drinfeld cusp form of weight $q^j + 1$ and type 1.

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4. Properly speaking, to call these spaces “spaces of Drinfeld modular forms” is an abuse of language; these spaces are just generated by the $u$-expansions associated to such forms, but since we will work here with formal series in $u$ only, it looked advantageous to make the identification between forms and formal series. We will do the same for Drinfeld quasi-modular forms; see [2] for further explanations.

5. This occurs if and only if $\tilde{M}_{w,m}^{\leq \ell} \neq \{0\}$, that is, if and only if $w \equiv 2m \pmod{q-1}$ with $w, l \geq 0$, it is unique up to multiplication by an element of $K^\times := K \setminus \{0\}$. 

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Let us write

\[ B_k(t) := \prod_{0 \leq i < j < k} (t^{q^i} - t^{q^j}) \in \mathbb{F}_q[t]. \]

For \( k \geq 2 \) and \( j \in \mathbb{Z} \), we then define \( E_{j,k} \) with the following determinant of Hankel type:

\[
E_{j,k} = \frac{1}{B_k(\theta)} \begin{vmatrix}
E_{j,1} & E_{j+1,1} & \cdots & E_{j+k-1,1} \\
E_{j-1,1} & E_{j,1} & \cdots & E_{j+k-2,1} \\
E_{j-2,1} & E_{j-1,1} & \cdots & E_{j+k-3,1} \\
\vdots & \vdots & \ddots & \vdots \\
E_{j-k+1,1} & E_{j-k+2,1} & \cdots & E_{j,1}
\end{vmatrix}.
\]

We shall show:

**Theorem 1** The following properties hold, for \( j \geq 0 \) and \( k \geq 1 \).

1. There exists a constant \( C(q,k) \) and a sequence of integers \((l_k)_{k \geq 1}\) such that for all \( j \geq C(q,k) \),

\[ E_{j,k} \in \tilde{M}_{(q^{k-1})/(q-1)} \setminus \tilde{M}_{(q^{k-1})/(q-1), k} \]

with \( l_k \to \infty \) for \( k \to \infty \).

2. For all \( j,k \) with \( j \geq 0 \), we have \( \text{ord}_{u=0} E_{j,k} = q^j(q^{2k} - 1)/(q^2 - 1) \).

3. For all \( j,k \) with \( j \geq 0 \), we have \( E_{j,k} \in A[[u]] \) and \( E_{j,k} \) is normalised.

4. For \( k = 1 \) and for \( k = 2 \) if \( q \geq 3 \), we have \( E_{j,k} = f_{j(q^{k-1} - 1)/(q-1), (q^{k-1})/(q-1), k} \) for all \( j \geq 0 \).

The interest of the theorem above is that it provides in an explicit way a family of normalised Drinfeld quasi-modular forms parametrised by \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \), with unbounded depths and weights, with high order of vanishing at \( u = 0 \), and with \( u \)-expansions defined over \( A \). The theorem gives a partial answer to the analogue of the Question above. Indeed, denoting by \( E_{l,m} \) the set whose elements are the weights \( w \) such that \( f_{l,w,m} \) is defined over \( A \), we have the following obvious consequence of Theorem 1.

**Corollary 2** If \( l = 1 \) and for any value of \( q \), or if \( q \geq 3 \) and \( l = q + 1 \), we have that \( f_{l,l(q^{q+1} - 1)} \in A[[u]] \) for all \( j \geq 0 \). Therefore, for the selected values of \( q,l,m \), the set \( E_{l,m} \) has infinitely many elements.

It can be shown that for \( k > 2 \), the degree of \( E_{j,k} \) in \( E \) is not maximal (it is maximal only for \( k = 1,2 \)), which may mean that for such values, \( E_{j,k} \) is not extremal. However, the fact that \( l_k \to \infty \) suggests that no natural threshold for the depth (as \( l = 4 \) in the classical case, as suggested by \cite{[8]} Conjecture 2]) exists in the Drinfeldian framework. Moreover, the presence of infinitely many \( f_{l,w,m} \)’s defined over \( A \) detected by Theorem 1 suggests that the \( A \)-algebra \( \tilde{M}_A \) generated by the Drinfeld quasi-modular forms with \( u \)-expansions defined over \( A \) could have, as a minimal set of generators, the \( f_{l,w,m} \)’s with \( w \in E_{l,m} \) for all \( l,m \)’s.

**Remark.** With the help of a formula appearing in \cite{[12]}, it is possible to explicitly compute the \( u \)-expansions of \( E_{j,k} \) for \( j \geq 0 \): we have \( E_{j,1} = \sum_{a \in A^+} a^{q^j} u_a \) with the notations of loc. cit. These forms, which are Hecke eigenforms, are also object of investigations by A. Petrov (private communication).
2 Determinants of Hankel’s type

We consider a field $L$, for $i,j \in \mathbb{Z}$, indeterminates $a_{i,j}$ and, over the polynomial ring $L[(a_{i,j})_{i,j \in \mathbb{Z}}]$, the shift operator $\sigma$ defined by $\sigma(a_{i,j}) = a_{i,j-1}$. Let us set:

$$H_n^{(k)} = \det((a_{n+i+j,j})_{0 \leq i,j \leq k-1}).$$

**Lemma 3** We have the following formula:

$$(\sigma H_n^{(k)})_{n+2}^{(k)} - (\sigma H_n^{(k+1)})_{n+2}^{(k-1)} = (\sigma H_{n+1}^{(k)})_{n+1}^{(k)}. \quad (1)$$

**Proof.** This is easy and left to the reader (see [5, Exercise 1 p. A IV.85 and Exercise 10 p. III.193]).

Let $(K, \tau)$ be an *inversive difference field*, that is the datum of a field $K$ together with an automorphism $\tau$ that will be supposed of infinite order in all the following. Let $\mathcal{G} = (\mathcal{G}_k)_{k \in \mathbb{Z}}$ be a sequence of $K$. We say, following [11], that $\mathcal{G}$ is a $\tau$-linear recurrent sequence if there exist $a_0, \ldots, a_s \in K$, not all vanishing, such that, for all $k$:

$$a_0 \mathcal{G}_k + a_1 (\tau \mathcal{G}_{k-1}) + \cdots + a_s (\tau^s \mathcal{G}_{k-s}) = 0.$$

We also say that $\mathcal{G}$ is a $\tau$-linearised recurrent sequence if there exist $a'_0, \ldots, a'_r \in K$, not all zero, such that for all $k$:

$$(\tau^k a'_0) \mathcal{G}_k + (\tau^k a'_1) \mathcal{G}_{k-1} + \cdots + (\tau^k a'_r) \mathcal{G}_{k-r} = 0.$$

Applying $\tau^{-k}$ to the identity above, one sees that $\mathcal{G}$ is a $\tau^{-1}$-linear recurrent sequence. Let us associate to $\mathcal{G}$ the following infinite matrix of “Hankel type”:

$$\mathcal{H}_\tau^{\infty}(\mathcal{G}) = \begin{pmatrix} \vdots & \vdots & \vdots \\ \cdots & \tau \mathcal{G}_{-1} & \mathcal{G}_0 & \tau^{-1} \mathcal{G}_1 & \cdots \\ \cdots & \tau \mathcal{G}_0 & \mathcal{G}_1 & \tau^{-1} \mathcal{G}_2 & \cdots \\ \cdots & \tau \mathcal{G}_1 & \mathcal{G}_2 & \tau^{-1} \mathcal{G}_3 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

We then have the elementary Proposition below, motivating the introduction of these determinants.

**Proposition 4** The following three conditions are equivalent.

1. $\mathcal{G}$ is a $\tau$-linear recurrent sequence.
2. $\mathcal{G}$ is a $\tau$-linearised recurrent sequence.
3. The matrix $\mathcal{H}_\tau^{\infty}(\mathcal{G})$ has finite rank.
Sketch of proof. If \( \mathcal{G} \) is \( \tau \)-linear recurrent so it satisfies, for some \( s \), relations
\[
a_0 G_k + a_1(\tau G_{k-1}) + \cdots + a_s(\tau^s G_{k-s}) = 0
\]
for all \( k \) with \( a_0, \ldots, a_s \) not all zero, then any \( s + 1 \) consecutive columns of the matrix \( H^\infty(\mathcal{G}) \) are linearly dependent.

Similarly, if \( \mathcal{G} \) is \( \tau \)-linearised recurrent then \( \mathcal{G}' = (\tau^{-k} G_k)_{k \in \mathbb{Z}} \) is \( \tau^{-1} \)-linear recurrent and we have relations
\[
a'_0(\tau^{-k} G_k) + a'_1(\tau^{-k} G_{k-1}) + \cdots + a'_s(\tau^{-k} G_{k-s}) = 0
\]
for all \( k \) with \( a'_0, \ldots, a'_s \) not all zero. This implies that any \( r + 1 \) consecutive rows of \( H^\infty(\mathcal{G}) \) are linearly dependent. Hence, in Proposition 4, 1 implies 3 and 2 implies 3.

Assuming now that 3 holds so that \( H^\infty(\mathcal{G}) \) has finite rank, the proof that \( \mathcal{G} \) is \( \tau \)-linear recurrent is easily obtained from Lemma 3 and the hints given in Exercise 1 p. A IV.85 by setting, for \( i, j \in \mathbb{Z}, a_{i,j} = \tau^{-j} G_{i+j} \), so that \( \sigma a_{i,j} = \tau a_{i,j} \) for all \( i, j \). The proof that 3 implies \( \mathcal{G} \) \( \tau \)-linearised recurrent is similar.

Proposition 4 motivates the study of the minors of \( H^\infty(\mathcal{G}) \) for a given sequence \( \mathcal{G} \). So far, the difference field \( K \) was unspecified and we need now to choose it to serve our purposes. Consider two supplementary indeterminates \( t, u \) and the field of formal series
\[
R = K((t))(\langle u \rangle).
\]
The Frobenius \( \mathbb{F}_q \)-linear endomorphism \( F \) of \( R \) splits as a product
\[
F = \chi \tau = \tau \chi,
\]
where \( \chi, \tau : R \to R \) are respectively \( K((u)) \)- and \( K((t)) \)-linear, uniquely determined by \( \chi(t) = t^q \) and \( \tau(u) = u^q \). There exists an extension \( K/R \) and an extension of \( \tau \) to \( K \) so that the so-obtained field \( K \) is perfect and the difference field \( (K, \tau) \) is inversive. Thus, we can extend \( \chi \) to \( K \) by setting \( \chi = F^{\tau^{-1}} \) and we notice on the way that the difference field \( (K, \chi) \) is also inversive.

For \( f \in K \), we introduce the following sequence of determinants which corresponds to a sequence of minors of \( H^\infty(\mathcal{G}) \), with \( \mathcal{G} = (\chi^{-k} f)_{k \in \mathbb{Z}} \):
\[
H_k(f) = \begin{vmatrix}
f & \tau f & \tau^2 f & \cdots & \tau^{k-1} f \\
\chi f & \chi \tau f & \chi \tau^2 f & \cdots & \chi \tau^{k-1} f \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\chi^{k-1} f & \chi^{k-1} \tau f & \chi^{k-1} \tau^2 f & \cdots & \chi^{k-1} \tau^{k-1} f
\end{vmatrix}
\]
For all \( k \), \( H_k(f) \) can be rewritten, thanks to the identity \( \chi = F^{\tau^{-1}} \), as
\[
H_k(f) = \begin{vmatrix}
f & \tau f & \tau^2 f & \cdots & \tau^{k-1} f \\
(\tau^{-1} f)^q & \tau^q f & \tau^2 f & \cdots & \tau^{k-1} f \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(\tau^{1-k} f)^q & (\tau^{2-k} f)^q & (\tau^{3-k} f)^q & \cdots & (\tau^{k-1} f)^q
\end{vmatrix}.
\]
(2)
Let \( K^\chi \) be the constant subfield of \( (K, \chi) \) and consider \( f \in K \).
Lemma 5 We have $H_k(f) \neq 0$ for all $k \geq 1$ if and only if $f$ is transcendental over $K\chi$.

Proof. Let $k$ be such that $H_k(f) = 0$. By proposition 4 this is equivalent to $G_j = (\chi^{-j}f)_j \in Z$ being $\tau$-linear recurrent. In other words, there exist $a_0, \ldots, a_s \in K$, not all zero, such that

$$a_0\chi^{-j}f + a_1\tau\chi^{-j}f + \cdots + a_s\tau^s\chi^{-j}f = 0, \quad j \in \mathbb{Z}.$$  

The latter identities are equivalent to:

$$(\chi^ja_0)f + (\chi^ja_1)f^q + \cdots + (\chi^ja_s)f^{q^r} = 0,$$

for all $j \in \mathbb{Z}$. A simple induction over $s \geq 1$ allows then to show that in fact, $H_k(f) = 0$ if and only if there exists $r > 0$ and $a_0, \ldots, a_r \in K\chi$ such that

$$a_0f + a_1f^q + \cdots + a_rf^{q^r} = 0,$$

and this amounts to the algebraicity of $f$ over $K\chi$. 

Remark. By Lemma 3 writing $H_{s,k}$ at the place of $\tau^sH_k(f)$ for a better display, we easily obtain the formula:

$$H_{s,k}^{q+1} - H_{s,k-1}^{q+1}H_{s,k+1} = H_{s-1,k}H_{s+1,k}, \quad (s \in \mathbb{Z}, k \geq 2).$$  

Formula (3) plays a role for $(\tau, \chi)$-difference fields similar to that of Sylvester's formula expressing determinants $\left| \frac{\partial^{i+j}f}{\partial z_1^{i}z_2^{j}} \right|_{0 \leq i,j \leq k-1}$ as in [1].

The elements $f = \sum_{i,j}c_{i,j}t^iu^j$ that we choose are either $d$, either $E = -(\theta q)^{-1}$, where $d$ is the unique solution (cf. [10]) in $F_q[t, \theta][[u]] \subset A[[\theta]][[u]]$ of the linear $\tau$-difference equation

$$(t - \theta q)\Delta(\tau^2X) + g(\tau X) - X = 0,\quad (4)$$

with $c_{0,0} = 1$ and $c_{i,0} = 0$ for $i > 0$, where $\Delta = -(\theta q)^{-1} \in A[[u]]$ is the opposite of the unique normalised cusp form of weight $q^2 - 1$ for $GL_2(A)$. We point out that in [10] we have computed some coefficients of the $u$-expansion of $d$. See also Lemma [10].

The relationship between $H_k(d)$ and $H_k(E)$ is simple. Since $\chi h = h$, we have

$$\chi^{i-1}\tau^{j-1}(E) = -(\theta q)^{-1}\tau(\chi^{i-1}\tau^{j-1}(d)) \quad (1 \leq i,j \leq k),$$

hence

$$H_k(E) = (-1)^k \frac{h^{1+q^2+\cdots+q^{k-1}}\tau(H_k(d))}{h^{1+q^2+\cdots+q^{k-1}}\tau(H_k(d))}.$$  

Lemma 6 We have, for $j \in \mathbb{Z}$ and $k \geq 1$:

$$E_{j,k} = \left. \frac{\tau^jH_k(E)}{B_k} \right|_{t = \theta}.$$  

Proof. It is proved in [10] that $E|_{t = \theta} = E = E_{0,1}$ from which it easily follows that $(\tau E)|_{t = \theta} = E_{1,1}$ and, by induction, that $(\tau^j E)|_{t = \theta}$, well defined for all $j \in \mathbb{Z}$, is equal to $E_{j,1}$. Comparing the definition of $E_{j,k}$ with (2) we immediately recover that the forms $E_{j,k}$ of Theorem 5 are, for $j \geq 0$, precisely the formal series of $K[[u]]$ obtained by substituting $t$ with $\theta$, a licit operation.
3 Properties of the determinants $H_k(d)$

By Lemma $[3]$ $H_k(d)$ are non-vanishing for all $k \geq 1$. Indeed, the $u$-expansion of $d$ having coefficients depending on $t$, we have that $d$ does not belong to $K((u))^{alg}$, which contains a copy of $K^x$ and the non-vanishing of $E$ then follows from $[5]$. Therefore, there exists an integer $\nu_k \geq 0$ such that

$$H_k(d) = \sum_{s \geq \nu_k} \kappa_{k,s} u^s$$

with $\kappa_{k,s} \in \mathbb{F}_q[t, \theta]$ and $\kappa_{k,\nu_k} \neq 0$. We will prove the Theorem below, from which we will deduce Theorem $[1]$.

**Theorem 7** The following properties hold for all $k \geq 1$.

1. $\nu_k = \frac{(s^k-1)(q^{k-1}-1)}{q^k-1}$,
2. $\kappa_{k,\nu_k} = B_k(t)$,
3. $H_k(d)/\kappa_{k,\nu_k}$ lies in $\mathbb{F}_q[t, \theta][[u^{q-1}]]$ and is normalised.

This section is devoted to the proof of Theorem $7$. Since the $u$-expansions of many forms involved (like $g, \Delta, d \ldots$), are actually expansions in powers of $u^{q-1}$, it will be convenient to set $v := u^{q-1}$.

In Section $3.1$, we first prove a general divisibility property for the coefficients of the $u$-expansion of $H_k(f)$ for formal series $f \in \mathbb{F}_q[t, \theta][[v]]$. Then, in Section $3.2$, we carefully study the growth of the degree in $t$ of the coefficients of $d$. Finally, we complete the proof of Theorem $7$ in Section $3.3$.

3.1 Computation of normalisation factors

**Proposition 8** Let $f$ be a formal series in $\mathbb{F}_q[t, \theta][[v]]$, so that we have a formal series expansion $H_k(f) = \sum_{s \geq 0} \kappa_s v^s$ with $\kappa_s \in \mathbb{F}_q[t, \theta]$ for all $s$. Then, the polynomial $B_k(t)$ divides $\kappa_s$ for all $s \geq 0$.

**Proof.** We observe that if for $1 \leq i, j \leq k$ we have formal expressions $f_{i,j} = \sum_{s \in \mathbb{N}} c_{i,j,s}$, then, by multilinearity:

$$\begin{vmatrix} f_{1,1} & \cdots & f_{1,k} \\ \vdots & \ddots & \vdots \\ f_{k,1} & \cdots & f_{k,k} \end{vmatrix}_{s_1, \ldots, s_k \in \mathbb{Z}} = \sum_{s_1, \ldots, s_k \in \mathbb{Z}} \begin{vmatrix} c_{1,1,s_1} & \cdots & c_{1,k,s_k} \\ \vdots & \ddots & \vdots \\ c_{k,1,s_1} & \cdots & c_{k,k,s_k} \end{vmatrix}_{s_1, \ldots, s_k \in I}.$$ 

(6)

Let us write $f = \sum_{s \geq 0} c_s v^s$ with $c_s \in \mathbb{F}_q[t, \theta]$. We set

$$f_{i,j} = \chi^{i-1} r_{i-1}^{j-1}(f) = \sum_{s \geq 0} \chi^{i-1} r_{i-1}^{j-1}(c_s v^s) = \sum_{s \geq 0} c_s (t^{q_i^{-1}}, \theta^{q_j^{-1}}) u^{q_i^{-1}s}$$

so that $c_{i,j,s} = c_s (t^{q_i^{-1}}, \theta^{q_j^{-1}}) u^{q_i^{-1}s}$. By $[4]$, we obtain that

$$H_k(f) = \sum_{s_1, s_2, \ldots, s_k} v^{s_1 + s_2 q + \cdots + s_k q^{k-1}} d_{s_1, s_2, \ldots, s_k},$$

(7)
where
\[
d_{s_1, s_2, \ldots, s_k} = \begin{vmatrix}
c_{s_1}(t, \theta) & c_{s_2}(t, \theta^q) & \cdots & c_{s_k}(t, \theta^{q^{k-1}}) \\
c_{s_1}(t^q, \theta) & c_{s_2}(t^q, \theta^q) & \cdots & c_{s_k}(t^q, \theta^{q^{k-1}}) \\
\vdots & \vdots & \ddots & \vdots \\
c_{s_1}(t^{q^{k-1}}, \theta) & c_{s_2}(t^{q^{k-1}}, \theta^q) & \cdots & c_{s_k}(t^{q^{k-1}}, \theta^{q^{k-1}})
\end{vmatrix}. \tag{8}
\]

We use the fact that \( c_{s} = \sum_{\mu} \kappa_{\mu, s} \theta^\mu \in \mathbb{F}_q[t, \theta], \) with \( \kappa_{\mu, s} \in \mathbb{F}_q[t] \). Let us apply (6) again, this time with \( f_{i,j} = \chi^{i-1} \tau^{j-1} c_{s_j} = \sum_{\mu} (\chi^{i-1} \kappa_{\mu, s_j}) \theta^{\mu q^j} \) and \( c_{i,j,\mu} = (\chi^{i-1} \kappa_{\mu, s_j}) \theta^{\mu q^j} \).

We obtain that
\[
d_{s_1, s_2, \ldots, s_k} = \sum_{\mu_1, \ldots, \mu_k} \theta^{\mu_1 + \mu_2 q + \cdots + \mu_k q^{k-1}} e_{\mu_1, \ldots, \mu_k},
\]
where
\[
e_{\mu_1, \ldots, \mu_k} = \begin{vmatrix}
\eta_1 & \cdots & \eta_k \\
\vdots & \ddots & \vdots \\
\chi^{k-1} \eta_1 & \cdots & \chi^{k-1} \eta_k
\end{vmatrix},
\]
with \( \eta_j = \kappa_{\mu_j, s_j} \). Now, by multilinearity, \( e_{\mu_1, \ldots, \mu_k} \) is a sum of Moore's determinants:
\[
M(\nu_1, \ldots, \nu_k) = \begin{vmatrix}
\nu_1 & \cdots & \nu_k \\
\vdots & \ddots & \vdots \\
\nu_1 q^{k-1} & \cdots & \nu_k q^{k-1}
\end{vmatrix}.
\]

We then apply the following lemma, which completes the proof of Proposition 8.

**Lemma 9** The formula
\[
M(0, 1, \ldots, k - 1) = B_k(t)
\]
holds. Moreover, for any choice of \( \nu_1, \ldots, \nu_k \), \( B_k(t) \) divides \( M(\nu_1, \ldots, \nu_k) \).

**Proof.** The explicit formula is a well known application, either of Moore's determinants, or Vandermonde's determinants. As for the divisibility property, this follows from an old and well known result of Mitchell, \[9\], as \( M(\nu_1, \ldots, \nu_k) \) can be viewed as a generalised Vandermonde's determinant. \( \Box \)

### 3.2 The degree of the coefficients of \( d \)

To prove Theorem 7, we will need a precise estimate of the growth of the degree in \( t \) of the coefficients of \( d \). Recall that the function \( d \) lies in \( \mathbb{F}_q[t, \theta][[v]] \), where \( v = u^{q-1} \). We will write in what follows
\[
d = \sum_{s \geq 0} c_s v^s, \tag{9}
\]
where \( c_s \in A[t] \). The aim of this section is to prove the following lemma, which is a slightly improved version of the second part of \[10\] Lemma 8.
Lemma 10 Let \( s \geq 0 \) and \( l \geq 0 \) be integers satisfying
\[ s < 1 + q^2 + \cdots + q^{2l}. \]
Then
\[ \deg_i c_s \leq l. \]
Moreover, for all \( l \geq 0 \) we have
\[ c_{1+q^2+\cdots+q^{2(l-1)}}(t) = (-1)^l t^l + \cdots, \]
where the dots stand for terms of degree \( < l \).

Remark. We have used here the convention that the empty sum is zero, so we have \( 1 + q^2 + \cdots + q^{2(l-1)} = 0 \) when \( l = 0 \).

Proof. Write
\[ g = 1 - [1]v + \cdots = \sum_{s \geq 0} \gamma_s v^s \in A[[v]], \]
and
\[ \Delta = -v(1 - v^{q-1} + \cdots) = \sum_{s \geq 0} \delta_s v^s \in vA[[v]]. \]
As in [10], we will use the following recursion formula for the coefficients \( c_s \), which easily follows from the \( \tau \)-difference equation (4) (see [10, Formula (30)]):
\[ c_s = \sum_{i+jq=s} \gamma_i(\tau c_j) + (t - \theta^q) \sum_{i+jq^2=s} \delta_i(\tau^2 c_j). \]  
(11)

We first prove by induction on \( s \geq 0 \) that \( \deg_i c_s \leq l \) for all \( l \) satisfying \( 1 + q^2 + \cdots + q^{2l} > s \). This statement is clearly true for \( s = 0 \) and \( s = 1 \), since \( c_0 = 1 \) and \( c_1 = -(t - \theta) \). Let now \( s \geq 2 \) and \( l \geq 0 \) be such that \( s < 1 + q^2 + \cdots + q^{2l} \), and consider the formula (11). If \( j \) is an index occurring in the first sum, then we have \( j \leq s/q < s \), hence
\[ \deg_i \tau c_j = \deg_i c_j \leq l \]
(12)
by induction hypothesis. Let now \((i, j)\) be a pair of indices occurring in the second sum. If \( i = 0 \), then \( \delta_i = 0 \) and \( \delta_i(\tau^2 c_j) = 0 \). If \( i \geq 1 \), then \( j \leq (s-1)/q^2 < 1 + \cdots + q^{2l-1} \), so
\[ \deg_i \tau^2 c_j = \deg_i c_j \leq l - 1 \]
(13)
by induction hypothesis applied to \( j \) and \( l - 1 \) (note that \( j < s \)). Since the coefficients \( \gamma_i \) and \( \delta_i \) do not depend on \( t \), it follows from (12), (13) and (11) that \( \deg_i c_s \leq l \) as required.

Let us now prove the second part of the lemma. We argue by induction on \( l \). For \( l = 0 \) and \( l = 1 \) the assertion is true. Let now \( l \geq 2 \) be an integer, and suppose that the formula (10) holds for \( l - 1 \). Put \( s := 1 + \cdots + q^{2(l-1)} \), and consider again the recursion formula (11). If \( j \) is any index appearing in the first sum, then, as before, \( j < s = 1 + \cdots + q^{2(l-1)} \). Hence \( \deg_i \tau c_j = \deg_i c_j \leq l - 1 \) by the first part of the lemma. Let us now consider a pair \((i, j)\) appearing in the second sum of (11). The smallest possible value for \( i \) is \( i = 1 \) (since \( s \equiv 1 \) (mod \( q^2 \))), for which we have \( j = 1 + \cdots + q^{2l-2} \). In this case, the induction hypothesis yields (since \( \delta_1 = -1 \))
\[ \delta_i(\tau^2 c_j) = (-1)^l t^{l-1} + \cdots \]
If now \( i > 1 \), then \( j < 1 + \cdots + q^{2(l-2)} \), hence \( \deg_i(\tau^2c_j) = \deg_c c_j \leq l - 2 \) by the first part of the lemma. It follows from these considerations that

\[
(t - \theta^q) \sum_{i+jq^2=s} \delta_i(\tau^2c_j) = (t - \theta^q)((-1)^l t^{l-1} + \cdots) = (-1)^l t^l + \cdots
\]

Summing up, we have proved that \( c_1 + q^2 + \cdots + q^{2(l-1)}(t) = (-1)^l t^l + \cdots \)

\[\square\]

### 3.3 Proof of Theorem 7

We can now begin the proof of Theorem 7. We write as before

\[
d = \sum_{s \geq 0} c_s v^s,
\]

where \( c_s \in A[t] \). It will be convenient to introduce the following notation. If \( s = (s_1, \ldots, s_k) \in \mathbb{N}^k \) (where \( \mathbb{N} = \{0, 1, \ldots\} \)), we define

\[
||s|| := \sum_{i=1}^k s_i q^{i-1}
\]

and, as in Section 3.1, Equation (8),

\[
d_s := \det(\chi^{i-1}\tau^{j-1}c_{s_j})_{1 \leq i, j \leq k} = \det(C_{s_1}, C_{s_2}, \ldots, C_{s_k}),
\]

where \( C_{s_j} \) is the column vector defined by

\[
C_{s_j} = \begin{pmatrix} \tau^{j-1}c_{s_j} \\ \chi(\tau^{j-1}c_{s_j}) \\ \vdots \\ \chi^{k-1}(\tau^{j-1}c_{s_j}) \end{pmatrix}.
\]

With this notation, the formula (7) writes

\[
H_k(d) = \sum_s d_s v^{||s||},
\]

where \( s \) runs over all \( k \)-tuples of \( \mathbb{N}^k \). To prove Theorem 7 we will show that the first non zero coefficient in the \( v \)-expansion (14) is obtained for only one multi-index \( s \), namely for

\[
s_0 := (1 + q^2 + \cdots + q^{2(k-2)}, \ldots, 1 + q^2, 1, 0).
\]

This will easily yield the theorem. We will need for this three lemmas.

**Lemma 11** Set \( s_0 := (1 + q^2 + \cdots + q^{2(k-2)}, \ldots, 1, 0) \in \mathbb{N}^k \). Then we have

\[
||s_0|| = \frac{(q^k - 1)(q^{k-1} - 1)}{(q^2 - 1)(q - 1)}
\]

and

\[
d_{s_0} = B_k(t).
\]
Proof. The first part of the lemma amounts to compute the double sum
\[ \sum_{i=1}^{k} \sum_{j=0}^{k-1-i} q^j q^{i-1}, \]
which is an exercise left to the reader.

To prove the second part, we use Lemma 10, Equality (10):
\[
d_{s_0} = \begin{vmatrix}
-1 & \cdots & -1 & t & \cdots & t^{k-2} & 1 \\
-1 & \cdots & -1 & t & \cdots & t^{k-2} & 1 \\
\vdots & & \vdots & \vdots & & \vdots & \\
-1 & \cdots & -1 & t & \cdots & t^{k-2} & 1
\end{vmatrix}
\]
Let us denote by \( C_1, \ldots, C_k \) the columns of this matrix. If we subtract \( q^{k-2} C_k \) to \( C_k \), then we eliminate the constant terms in \( C_{k-1} \), that is, we get the new penultimate column \( C_k' \).

Repeating this process for the columns \( C_j, j = k-3, \ldots, 1 \), we see by induction that
\[
d_{s_0} = \begin{vmatrix}
-1 & \cdots & -1 & t & \cdots & t^{k-2} & 1 \\
-1 & \cdots & -1 & t & \cdots & t^{k-2} & 1 \\
\vdots & & \vdots & \vdots & & \vdots & \\
-1 & \cdots & -1 & t & \cdots & t^{k-2} & 1
\end{vmatrix}
\]
Now, this determinant is equal to
\[
\begin{vmatrix}
1 & t & \cdots & t^{k-1} \\
1 & t^q & \cdots & t^{(k-1)q} \\
\vdots & \vdots & & \vdots \\
1 & t^{q^{k-1}} & \cdots & t^{(k-1)q^{k-1}}
\end{vmatrix}
\]
which is equal to \( B_k(t) \) (Vandermonde determinant; see also Lemma 9).

The next lemma roughly says that if a coefficient \( d_s \) is not zero in (14), and if we reorder the coefficients \( c_{s_i} \) such that the sequence \( (\deg_t c_{s_i})_i \) is non decreasing, then the degrees \( \deg_t c_{s_i} \) grow at least linearly in \( i \).

Lemma 12 Let \( d = (s_1, \ldots, s_k) \in \mathbb{N}^k \) such that \( d_s \neq 0 \). Let \((i_1, \ldots, i_k)\) be a permutation of the set \( \{1, \ldots, k\} \) such that
\[
\deg_t c_{s_{i_1}} \leq \cdots \leq \deg_t c_{s_{i_k}}.
\]
Then, for all \( l \), we have
\[
\deg_t c_{s_{i_l}} \geq l - 1.
\]
Proof. Let us write $d_{s} = \det(C_{s1}, \ldots, C_{sk})$. Suppose that there exists an $l$ such that $\deg_t c_{s_{ij}} \leq l - 2$. Then, since the operator $\tau$ does not change the degree in $t$, the family $(c_{s_{ij}}, \ldots, \tau^{l-1} c_{s_{ij}})$ consists of $l$ polynomials in $K[t]$ of degree $\leq l - 2$, so they are linearly dependent over $K$. Hence there exist elements $\lambda_j \in K$, not all zero, such that

$$
\sum_{j=1}^{l} \lambda_j \tau^{l-1} c_{s_{ij}} = 0.
$$

If we now apply the operator $\chi^{i-1}$ ($1 \leq i \leq k$), we find :

$$
\sum_{j=1}^{l} \lambda_j \chi^{i-1} \tau^{l-1} c_{s_{ij}} = 0 \quad (1 \leq i \leq k).
$$

In other words, we get $\sum_{j=1}^{l} \lambda_j C_{ij} = 0$, that is, a non trivial linear combination of the columns $(i_1, \ldots, i_l)$ in $d_{s}$. Hence $d_{s} = 0$, which is a contradiction. \hfill \square

We introduce a further notation. If $\sigma \in S_{\{1, \ldots, k\}}$ is a permutation of the set $\{1, \ldots, k\}$ and if $s = (s_1, \ldots, s_k)$ is an element of $\mathbb{N}^k$, we define $s^\sigma := (s_{\sigma(1)}, \ldots, s_{\sigma(k)})$. We recall that $s_0$ was defined in Lemma [11]

**Lemma 13** Let $\sigma$ be a permutation of the set $\{1, \ldots, k\}$ such that $\sigma \neq \text{Id}$. Then

$$
||s_0^\sigma|| > ||s_0||.
$$

**Proof.** We argue by induction on $k$. For $k = 1$ there is nothing to prove. Let now $k \geq 2$ be an integer and let $\sigma$ be a permutation as in the lemma. For $l \geq 1$, define $t_l$ by $t_l := 1 + \cdots + q^{2(l-2)}$. We will also use the notation $s_0^{(k)}$ instead of $s_0$ to indicate the dependence on $k$. Thus we have

$$
s_0 = s_0^{(k)} = (t_k, \ldots, t_1) \quad \text{and} \quad ||s_0^{(k)}|| = \sum_{l=1}^{k} t_l q^{k-l}.
$$

Let further $\tau$ denote the permutation of $\{1, \ldots, k\}$ such that $(s_0^{(k)})^\sigma = (t_{\tau(k)}, \ldots, t_{\tau(1)})$.

First, suppose that $\tau(k) = k$. Then $\tau$ induces a non trivial permutation of the set $\{1, \ldots, k-1\}$, and

$$
||(s_0^{(k)})^\sigma|| - ||s_0^{(k)}|| = \sum_{l=1}^{k-1} (t_{\tau(l)} - t_l) q^{k-l} = q ||(s_0^{(k-1)})^\sigma'|| - ||s_0^{(k-1)}||,
$$

where $\sigma'$ is the (non trivial) permutation of $\{1, \ldots, k-1\}$ such that $(s_0^{(k-1)})^\sigma' = (t_{\tau(k-1)}, \ldots, t_{\tau(1)})$. By induction hypothesis, it immediately follows that $||(s_0^{(k)})^\sigma|| - ||s_0^{(k)}|| > 0$.

Suppose now that $\tau(k) \neq k$. Then

$$
||(s_0^{(k)})^\sigma|| \geq t_k q^{k-\tau^{-1}(k)} \geq qt_k = \frac{q(1+q^{2(k-1)})}{q^2 - 1} > \frac{(q^k - 1)(q^{k-1} - 1)}{(q^2 - 1)(q - 1)},
$$

hence $||(s_0^{(k)})^\sigma|| > ||s_0^{(k)}||$ by Lemma [11]. \hfill \square

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Proof of Theorem 7. We define \( s_0 = (s_{0,1}, \ldots, s_{0,k}) \) as in Lemma 11. Thus we have
\[
s_{0,l} = 1 + \cdots + q^{2(k-1-l)} \quad (1 \leq l \leq k).
\]
Let now \( s = (s_1, \ldots, s_k) \in \mathbb{N}^k \) be such that \( d_s \neq 0 \). Choose a permutation \( \sigma \) of \( \{1, \ldots, k\} \) such that \( \deg_t c_{s_{\sigma(k)}} \leq \cdots \leq \deg_t c_{s_{\sigma(1)}} \). By Lemma 12 (note the different order that we have chosen here), we have \( \deg_t c_{s_{\sigma(l)}} \geq k - l \) for all \( l \). Hence, by Lemma 10,
\[
s_{\sigma(l)} \geq 1 + \cdots + q^{2(k-1-l)} = s_{0,l},
\]
or
\[
s_l \geq s_{0,\sigma^{-1}(l)}.
\]
(15)

It follows, by Lemma 13, that we have
\[
\|s\| \geq \|s_0^{-1}\| \geq \|s_0\|,
\]
and the equality \( \|s\| = \|s_0\| \) holds only if \( \sigma = \text{Id} \). In that case, the inequality (15) shows that \( \|s\| = \|s_0\| \) only if \( s = s_0 \). Thus, we have shown that in the \( v \)-expansion (14), the first non-zero coefficient is \( d_{s_0} \) :
\[
H_k(d) = d_{s_0} \|s_0\| \text{ higher terms}
\]

The points 1 and 2 of Theorem 7 follow at once from this and Lemma 11 (recall that \( v = u^{q-1} \), so \( \nu_k = (q-1)\|s_0\| \)). The point 3 is then a consequence of Proposition 8. \( \square \)

4 Proof of Theorem 1

In order to prove Theorem 1, we need to introduce a few notation. For any integer \( l \geq 0 \) and any triple \( (\mu, \nu, m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \), we denote by \( \widetilde{M}^{\leq l}_{\mu,\nu, m} \) the \( K((t)) \)-module of almost \( A \)-quasimodular forms of weight \( (\mu, \nu) \), type \( m \) and depth \( \leq l \) (see [10], Section 4.2), and we set
\[
\widetilde{M}^{\mu,\nu, m} = \bigcup_{l \geq 0} \widetilde{M}^{\leq l}_{\mu,\nu, m}.
\]

We will also write \( l(f) \) for the depth of the form \( f \). As in [10] \( \S 5.1 \), we further set \( h = hd \), and we finally define
\[
M^{\mu,\nu, m} = K((t))[g, h, \Delta^{-1}, E, h] \cap \widetilde{M}^{\mu,\nu, m}.
\]

We have :

Lemma 14 1. If \( f \in M^{\mu,\nu, m} \), then \( \tau f \in M^{\mu,\nu, m} \) and \( \chi f \in M^{\mu,\nu, m} \).

2. For all \( j \geq 0 \) we have
\[
l(\tau^j E) \leq 1, \quad l(\tau^j h) \leq 1, \quad l(\chi^j E) \leq q^j, \quad l(\chi^j h) \leq q^j.
\]

Proof. We first prove that the following equalities hold :
\[
\tau h = \Delta E, \quad \tau E = \frac{1}{t - \theta^q} (gE + h), \quad \chi h = (t - \theta)^q E^q - \frac{g}{\Delta} h^q, \quad \chi E = \frac{h^q}{\Delta}.
\]
(16)
The first equality follows at once from the definitions of \( h \) and \( E \) and the second is Lemma 23 of [10]. The last one then follows from the first:
\[
\chi E = \chi \left( \frac{\tau h}{\Delta} \right) = \frac{h^q}{\Delta}.
\]

Finally, to prove the third equality, we use the following one, which follows for instance from [10, Proposition 10] or [3, Proposition 2.7]:
\[
\chi = \frac{t - \theta^q}{\Delta^q} \tau (h^q) - \frac{g^q}{\Delta} \tau h.
\]

Applying \( \chi \) to both sides of this equality, and using the formula \( \tau h = \Delta E \), we get
\[
\chi h = \frac{t^q - \theta^q}{\Delta^q} \tau (h^q) - \frac{g^q}{\Delta} h^q = (t - \theta)^q E^q - \frac{g^q}{\Delta} h^q.
\]

The first part of the lemma follows at once from the relations (16) (we recall that \( E \in \tilde{M}^{\leq 0}_{q,1,1} \) and \( h \in \tilde{M}^{\leq 0}_{q,1,1} \)). The second part is a simple induction, noticing that the depth of a form \( f \) in \( M^\tau_{\nu,m} \) is nothing else than the degree \( \deg_E f \), when \( f \) is seen as an element of the polynomial ring \( K((t))((g,h,\Delta^{-1}))[E,h] \).

We now have all the elements to prove Theorem 1.

**Proof of Theorem 1** For all \( i,j \in \{1, \ldots, k\} \) we have, by Lemma [14]
\[
\chi^{i-1} \tau^{j-1} E \in \tilde{M}^{\leq q^{i-1}}_{q^j-1,q^j-1,1}.
\]

It follows, by a straightforward computation, that
\[
H_k(E) \in \tilde{M}^{\leq (q^k-1)/(q-1)}_{(q^k-1)/(q-1),(q^k-1)/(q-1),k}.
\]

Replacing \( t \) by \( \theta \), we then obtain the value of the weight and the type of \( E_{j,k} = (\tau^j H_k(E))_{|t=\theta} \).

We prove the last part of the first property of the Theorem asserting that the degree in \( E \) of \( E_{j,k} \) is not smaller than some integer \( l_k \) with \( l_k \to \infty \) as \( k \to \infty \).

By the main theorem of [10], if \( f \in M^\tau_{\nu,m} \) is non-zero and if
\[
w \geq 4l(2q(q + 2)(3 + 2q)l + 3(q^2 + 1))^{3/2},
\]
then
\[
\ord_w f \leq 16q^3(3 + 2q)^2lw.
\]

We can choose \( C(q,k) \) big enough so that if \( j \geq C(q,k) \), then (18) holds with \( f = E_{j,k}, w = (q^k - 1)(q^j + 1)/(q - 1) \) and \( l = \deg_E(E_{j,k}) \). Then, we get
\[
l \geq \frac{q^{3-3}}{1 + q^j} \frac{1}{16(1 + q)(3 + 2q)^2(1 + q^k)}
\]
so that, enlarging \( C(q,k) \) if necessary, we get, for \( j \geq C(q,k) \),
\[
l \geq \frac{1}{32(1 + q)(3 + 2q)^2(1 + q^k)}
\]
which gives the required property of growth of the sequence $(l_k)_k$.

Using now Theorem 7 and (5), we find, for all $j \geq 0$:

$$\frac{\tau_j H_k(E)}{\kappa_k,\nu_k} = (-1)^k h^q q^{-k-1} \tau_{j+1} (\kappa_{k,\nu_k} u^{\nu_k} + \cdots) = (-1)^k h^q q^{-k-1} (u^{q^{j+1}+\nu_k} + \cdots) \in A[[t, u]].$$

Substituting $t = \theta$ in this equality yields

$$E_{j,k} = (-1)^k h^q q^{-k-1} (u^{q^{j+1}+\nu_k} + \cdots) \in A[[u]].$$

The properties 2, 3 of Theorem 1 follow at once from this and from (17).

It remains to show the property 4. We consider first the case $k = 1$. By definition, $H_1(E) = E$ and $E_{j,1} = (\tau_j E)|_{t=\theta}$ with ord$_{u=0} E_{j,1} = q^j$. By [3, Theorem 1.2, Proposition 2.3], $E_{j,1}$ is proportional to the function $x_j$ defined there, and hence extremal. Moreover, it is normalised, so that $E_{j,1} = f_{1,q^{j+1}+1}$ for $j \geq 0$.

Let us assume now that $k = 2, q \geq 3$. In this case, by [3, Theorem 1.3, Proposition 2.13], we see that $E_{j,2}$ is proportional to the form $\xi_j$ defined there, and hence extremal. Since it is normalised and defined over $A$, the proof of Theorem 1 is complete. 

References


