Koszul calculus of preprojective algebras

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Plan

Koszul calculus of a quadratic quiver algebra A

Fix a finite quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ and a field $\mathbb{F}$.

The vertex space $k = \mathbb{F}\mathcal{Q}_0$ is a commutative ring by associating with $\mathcal{Q}_0$ a complete set of orthogonal idempotents $e_i,\ i \in \mathcal{Q}_0$.

The arrow space $V = \mathbb{F}\mathcal{Q}_1$ is a k-bimodule. The tensor k-algebra $T_k(V)$ of V is isomorphic to the path algebra $\mathbb{F}\mathcal{Q}$.

Let R be a sub-k-bimodule of $V \otimes_k V \cong \mathbb{F}\mathcal{Q}_2$. The associative k-algebra $A = T_k(V)/(R)$ is called a *quadratic algebra over* $\mathcal{Q}$ *and* $\mathbb{F}$. It is graded by the path length with $A_0 \cong k$ and $A_1 \cong V$.

In the one vertex case $k = \mathbb{F}$, the Koszul calculus was defined and studied in

R. Berger, T. Lambre, A. Solotar, Koszul calculus, *Glasg. Math. J.* 2018.

Defining the Koszul bimodule complex $K(A)$

For any $p \geq 0$, W_p denotes the sub-k-bimodule of $V^{\otimes_k p} \subseteq A^{\otimes_k p}$ defined by

$$W_p = \bigcap_{i+2+j=p} V^{\otimes_k i} \otimes_k R \otimes_k V^{\otimes_k j}.$$

$K(A)$ is the subcomplex of the bar resolution $B(A)$ defined by the sub-A-bimodules $A \otimes_k W_p \otimes_k A$ of $A \otimes_k A^{\otimes_k p} \otimes_k A$.

The differential d of $K(A)$ is given on $A \otimes_k W_p \otimes_k A$ by

$$d(a \otimes_k x_1 \ldots x_p \otimes_k a') = a x_1 \otimes_k x_2 \ldots x_p \otimes_k a' + (-1)^p a \otimes_k x_1 \ldots x_{p-1} \otimes_k x_p a'.$$

When $K(A)$ is a resolution of A, A is said to be Koszul.

Defining the Koszul calculus of A

For any A-bimodule M, we replace the bar resolution $B(A)$ by $K(A)$ in the definition of the Hochschild spaces $\mathrm{HH}_p(A, M)$ and $\mathrm{HH}^p(A, M)$, and we obtain the Koszul spaces $\mathrm{HK}_p(A, M)$ and $\mathrm{HK}^p(A, M)$.

The inclusion $K(A) \hookrightarrow B(A)$ induces linear maps

$$\mathrm{HK}_p(A, M) \to \mathrm{HH}_p(A, M),\ \mathrm{HH}^p(A, M) \to \mathrm{HK}^p(A, M)$$

which are always isomorphisms for $p = 0$ and $p = 1$, and if A is Koszul these maps are isomorphisms for any p.

We also need cup and cap products on Koszul cohomology and homology. They are defined by restricting the standard cup and cap products defined on Hochschild cochains and chains. For example the Koszul cup product is defined as follows.

Defining the Koszul calculus of A

Let P, Q be A-bimodules. For Koszul cochains $f \in \mathrm{Hom}_{k^e}(W_p, P)$ and $g \in \mathrm{Hom}_{k^e}(W_q, Q)$, $f \underset{K}{\smile} g \in \mathrm{Hom}_{k^e}(W_{p+q}, P \otimes_A Q)$ is given by

$$(f \underset{K}{\smile} g)(x_1 \dots x_{p+q}) = (-1)^{pq} f(x_1 \dots x_p) \otimes_A g(x_{p+1} \dots x_{p+q}).$$

Then $\tilde{A} = (\mathrm{Hom}_{k^e}(W_\bullet, A), b_K, \underset{K}{\smile})$ is a DG algebra and $\mathrm{Hom}_{k^e}(W_\bullet, M)$

is a DG $\tilde{A}$-bimodule for left and right actions $\underset{K}{\smile}$. Similarly $M \otimes_{k^e} W_\bullet$ is a DG $\tilde{A}$-bimodule for actions $\underset{K}{\frown}$.

The *Koszul calculus of A* consists of the graded associative algebra $(\mathrm{HK}^\bullet(A), \underset{K}{\smile})$ and of the graded $\mathrm{HK}^\bullet(A)$-bimodules $\mathrm{HK}^\bullet(A, M)$ and $\mathrm{HK}_\bullet(A, M)$ for all A-bimodules M. When $M = A$, the Koszul calculus is said to be *restricted*.

Preprojective algebras

Let Q be a quiver whose underlying graph is denoted by Δ. Define a quiver Q^* whose vertex set is Q_0 and whose arrow set is $Q_1^* = \{a^*; a \in Q_1\}$ where $\mathfrak{s}(a^*) = \mathfrak{t}(a)$ and $\mathfrak{t}(a^*) = \mathfrak{s}(a)$.

Let $\overline{Q}$ be the quiver whose vertex set is $\overline{Q}_0 = Q_0$ and whose arrow set is the disjoint union $\overline{Q}_1 = Q_1 \cup Q_1^*$.

Let $\mathbb{F}$ be a field. As before, $k = \mathbb{F}Q_0$, $V = \mathbb{F}\overline{Q}_1$, and $T_k(V) \cong \mathbb{F}\overline{Q}$.

The *preprojective algebra* $A(\Delta)$ is the quadratic k-algebra over $\overline{Q}$ defined by the quadratic relations

$$\sigma_i := \sum_{\substack{a \in Q_1 \\ \mathfrak{t}(a)=i}} aa^* - \sum_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} a^* a$$

for all $i \in Q_0$.

Why preprojective algebras are of interest for us ?

If a quadratic quiver algebra A is Koszul, its Koszul invariants are the same as its Hochschild invariants. If A is not Koszul, new invariants can be expected. The following is standard.

Proposition

Assume that the graph Δ is distinct from A_1 and A_2. The following are equivalent.
(i) Δ is Dynkin of type ADE.
(ii) $A(\Delta)$ is not Koszul.
(iii) $A(\Delta)$ is finite dimensional.

In our paper, we compute the restricted Koszul calculus of the ADE preprojective algebras A. From these computations, we prove that the inclusion $\mathrm{HH}^2(A) \hookrightarrow \mathrm{HK}^2(A)$ is not surjective, except in type E_8 with $\mathrm{char}(\mathbb{F}) = 2$.

A Poincaré Van den Bergh duality

In our paper, we prove the following.

Theorem

Let A be the preprojective algebra of a connected graph Δ distinct from A_1 and A_2, over a field $\mathbb{F}$. Let M be an A-bimodule.
(i) The complex $K(A)$ has length 2. In particular, $\mathrm{HK}^p(A, M) = \mathrm{HK}_p(A, M) = 0$ for any $p > 2$.
(ii) The $\mathrm{HK}^\bullet(A)$-bimodules $\mathrm{HK}^\bullet(A, M)$ and $\mathrm{HK}_{2-\bullet}(A, M)$ are isomorphic.

Consequently, the Koszul calculus in homology can be deduced from the Koszul calculus in cohomology.
Now we want to explain (i) and (ii) in more details.

(i) $K(A)$ has length 2

Since $W_p = (W_{p-1} \otimes_k V) \cap (V \otimes_k W_{p-1})$ for all $p > 2$ and $W_2 := R \neq 0$, it suffices to prove that

$$W_3 := (R \otimes_k V) \cap (V \otimes_k R) = 0.$$

We assume that $\Delta \neq \mathrm{A}_1$ and prove that $W_3 \neq 0$ implies $\Delta = \mathrm{A}_2$:

Let u be a non-zero element in W_3. We may assume that u is in eW_3f for some vertices e and f. Then u can be written uniquely in the basis of $R \otimes_k V$ and in the basis of $V \otimes_k R$. Comparing the two decompositions, we show that

$$e \neq f, \; Q_0 = \{e, f\}, \; |Q_1| = 1.$$

Thus $\Delta = \mathrm{A}_2$.

(ii) the duality $\mathrm{HK}^p(A, M) \cong \mathrm{HK}_{2-p}(A, M)$

This isomorphism comes from an explicit isomorphism from the complex defining $\mathrm{HK}^p(A, M)$ to the complex defining $\mathrm{HK}_{2-p}(A, M)$:

Theorem

Define $\omega_0 = \sum_i e_i \otimes \sigma_i$. It is a Koszul 2-cycle with coefficients in A. For each Koszul p-cochain f with coefficients in M, we define the Koszul $(2-p)$-chain $\theta_M(f)$ with coefficients in M by

$$\theta_M(f) = \omega_0 \underset{K}{\frown} f.$$

Then θ_M is an isomorphism of complexes.
Moreover, the equalities

$$\theta_{M\otimes_A N}(f \underset{K}{\smile} g) = \theta_M(f) \underset{K}{\frown} g = f \underset{K}{\frown} \theta_N(g)$$

hold for Koszul cochains f and g with coefficients in M and N.

A fundamental class for the duality

Passing the isomorphism of complexes

$$\theta_M = \omega_0 \underset{K}{\frown} - : \mathrm{Hom}_{k^e}(W_\bullet, M) \to M \otimes_{k^e} W_{2-\bullet}$$

to homology, we get a linear isomorphism

$$H(\theta_M) = \overline{\omega}_0 \underset{K}{\frown} - : \mathrm{HK}^\bullet(A, M) \to \mathrm{HK}_{2-\bullet}(A, M).$$

So the class $\overline{\omega}_0 \in \mathrm{HK}_2(A)$ is an analogue of the *fundamental class* used in the classical Poincaré's duality for singular (co)homology.

A generalisation of the 2-Calabi-Yau property

Take $M = A^e := A \otimes A^{op}$ in our duality and identify $A^e \otimes_{k^e} W_\bullet$ to $K(A)$. We obtain an isomorphism of complexes

$$\theta_{A^e} : \mathrm{Hom}_{k^e}(W_\bullet, A^e) \to K(A)[-2] \text{ in } A\text{-Bimod}$$

inducing an isomorphism

$$\mathrm{RHom}_{A^e}(K(A), A^e) \to K(A)[-2] \text{ in } \mathcal{D}^b(A\text{-Bimod}).$$

Then we say that A is *Koszul complex Calabi-Yau of dimension 2*.

Assume that Δ is not Dynkin ADE, so that A is Koszul. Then $K(A) \cong A$ in $\mathcal{D}^b(A\text{-Bimod})$, implying an isomorphism

$$\mathrm{RHom}_{A^e}(A, A^e) \to A[-2] \text{ in } \mathcal{D}^b(A\text{-Bimod}).$$

We recover that A is 2-Calabi-Yau in Ginzburg's sense.

The situation is new in the ADE types !

Passing to homology in 2-Kc-Calabi-Yau algebras, we get
(i) The A-bimodule $\mathrm{HK}^2(A, A^e)$ is isomorphic to the A-bimodule A.
(ii) $\mathrm{HK}^1(A, A^e) = 0$.
(iii) The A-bimodule $\mathrm{HK}^0(A, A^e)$ is isomorphic to the A-bimodule $H_2(K(A))$.

Passing to homology in 2-Calabi-Yau algebras, we get
(i) The A-bimodule $\mathrm{HH}^2(A, A^e)$ is isomorphic to the A-bimodule A.
(ii) $\mathrm{HH}^p(A, A^e) \cong 0$ for $p \neq 2$.

Assume that A is Dynkin ADE. Then $H_2(K(A)) \neq 0$ because A is not Koszul. Therefore, since $\mathrm{HH}^0(A, A^e) \cong \mathrm{HK}^0(A, A^e)$, A is not 2-Calabi-Yau in Ginzburg's sense. However, A is 2-Kc-Calabi-Yau !

An n-generalisation

Definition

Let $A = T_k(V)/(R)$ be a quadratic k-algebra over a finite quiver $\mathcal{Q}$. Let $n \geq 0$ be an integer. We say that A is Koszul complex Calabi-Yau of dimension n (n-Kc-Calabi-Yau) if
(i) the Koszul bimodule complex $K(A)$ of A has length n, and
(ii) $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in $\mathcal{D}^b(A\text{-Bimod})$.

Assume that A is Koszul. Then A is n-Kc-Calabi-Yau if and only if A is n-Calabi-Yau.

Conjecture

Let $A = T_k(V)/(R)$ be a quadratic k-algebra over a finite quiver $\mathcal{Q}$. If A is n-Calabi-Yau and n-Kc-Calabi-Yau, then A is Koszul (proved if $n \leq 3$).

Duality for n-Kc-Calabi-Yau algebras

Theorem

Let A be an n-Kc-Calabi-Yau algebra. Then for any A-bimodule M, the vector spaces $\mathrm{HK}^p(A, M)$ and $\mathrm{HK}_{n-p}(A, M)$ are isomorphic.

Sketch of proof : For any finitely generated projective A-bimodule P, we have a standard isomorphism

$$M \otimes_{A^e} \mathrm{Hom}_{A^e}(P, A^e) \cong \mathrm{Hom}_{A^e}(P, M) \text{ in } \mathsf{Vect}_{\mathbb{F}} \ \text{ inducing}$$

$$M \overset{L}{\otimes}_{A^e} \mathrm{RHom}_{A^e}(K(A), A^e) \cong \mathrm{RHom}_{A^e}(K(A), M) \text{ in } \mathcal{D}^b(\mathsf{Vect}_{\mathbb{F}}).$$

Combining with $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in $\mathcal{D}^b(A\text{-Bimod})$, we conclude

$$\mathrm{RHom}_{A^e}(K(A), M) \cong M \overset{L}{\otimes}_{A^e} K(A)[-n] \text{ in } \mathcal{D}^b(\mathsf{Vect}_{\mathbb{F}}).$$

Towards an enriched duality isomorphism

Definition

Let A be an n-Kc-Calabi-Yau algebra. The image $c \in \mathrm{HK}_n(A)$ of the unit 1 of A by the duality isomorphism $\mathrm{HK}^0(A) \cong \mathrm{HK}_n(A)$ is called the fundamental class of the n-Kc-Calabi-Yau algebra A.

We want to define strong Kc-Calabi-Yau algebras.

For this we need to work with DG $\tilde{A}$-bimodules in A-Bimod, where $\tilde{A}$ denotes the DG algebra $\mathrm{Hom}_{A^e}(K(A), A)$.

Following Yekutieli's book (Derived categories. Cambridge Studies in Advanced Mathematics 183, CUP, 2020), $\mathcal{C}(\tilde{A}, A\text{-Bimod})$ ($\mathcal{C}(\tilde{A}, \mathrm{Vect}_{\mathbb{F}})$) denotes the category of DG $\tilde{A}$-bimodules in A-Bimod (in $\mathrm{Vect}_{\mathbb{F}}$), and the associated bounded derived categories are defined.

Duality for strong generalised Calabi-Yau algebras

Definition

Let A be an n-Kc-Calabi-Yau algebra. Then A is said to be strong n-Kc-Calabi-Yau if the derived functor of the endofunctor $\mathrm{Hom}_{A^e}(-, A^e)$ *of* $\mathcal{C}^b(\tilde{A}, A\text{-Bimod})$ *exists and if* $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ *in* $\mathcal{D}^b(\tilde{A}, A\text{-Bimod})$.

Theorem

Let A be a strong n-Kc-Calabi-Yau algebra with fundamental class c. We assume that the derived functors of the functors $\mathrm{Hom}_{A^e}(-, A)$ *and* $A \otimes_{A^e} -$ *from* $\mathcal{C}^b(\tilde{A}, A\text{-Bimod})$ *to* $\mathcal{C}^b(\tilde{A}, \mathrm{Vect}_{\mathbb{F}})$ *exist.*
Then

$$c \underset{K}{\frown} - : \mathrm{HK}^{\bullet}(A) \to \mathrm{HK}_{n-\bullet}(A)$$

is an isomorphism of graded $\mathrm{HK}^{\bullet}(A)$*-bimodules.*

Dimension of the preprojective algebra

type of A	*Coxeter number*	$\dim(A)$
A_n, $n \geq 1$	$h = n + 1$	$\frac{n(n+1)(n+2)}{6}$
D_n, $n \geq 4$	$h = 2(n-1)$	$\frac{n(n-1)(2n-1)}{3}$
E_6	$h = 12$	156
E_7	$h = 18$	399
E_8	$h = 30$	1240

The formula $\dim(A) = \frac{h(h+1)n}{6}$ can be found in : A. Malkin, V. Ostrik, M. Vybornov, Quiver varieties and Lusztig's algebra, *Adv. Math.* 203 (2006) 514-536 (Corollary of Theorem 2.3). In this paper, the definition of the preprojective algebra is different from the standard definition we have used, but it is the same in ADE types.

All the non-Dynkin preprojective algebras are infinite-dimensional.

Dimension of the Koszul cohomology I

type of A	$\dim \mathrm{HK}^0(A)$	$\dim \mathrm{HK}^1(A)$	$\dim \mathrm{HK}^2(A)$	$\dim \mathrm{HH}^2(A)$
A_n, $n \geq 3$	$m_A + 1$	$n - m_A - 1$	n	$n - m_A - 1$
D_n, $n \geq 4$	if n even $n + m_D$	if char $\neq 2$ $n - m_D - 2$	if char $\neq 2$ n	if char $\neq 2$ 0,1 if n even,odd
	if n odd $n + m_D - 1$	if char $= 2$ $n - 2$	if char $= 2$ $n + m_D$	if char $= 2$ $n + m_D - 2$

where $m_A = \lfloor \frac{n-1}{2} \rfloor$, $m_D = \lfloor \frac{n-2}{2} \rfloor$, and char denotes the characteristic of the field $\mathbb{F}$.

Observation : the natural inclusion $\mathrm{HH}^2(A) \hookrightarrow \mathrm{HK}^2(A)$ is not surjective. In other words, there is more information in $\mathrm{HK}^2(A)$ than in $\mathrm{HH}^2(A)$.

Dimension of the Koszul cohomology II

type	$\mathrm{HK}^0(A)$	$\mathrm{HK}^1(A)$	$\mathrm{HK}^2(A)$	$\mathrm{HH}^2(A)$
E_6	5	3 if char $\neq 2,3$	6 if char $\neq 2,3$	2 if char $\neq 2,3$
		4 if char $= 2$	7 if char $= 2$	5 if char $= 2$
		4 if char $= 3$	7 if char $= 3$	3 if char $= 3$
E_7	10	3 if char $\neq 2,3$	7 if char $\neq 2,3$	0 if char $\neq 2,3$
		6 if char $= 2$	10 if char $= 2$	9 if char $= 2$
		4 if char $= 3$	8 if char $= 3$	1 if char $= 3$
E_8	12	4 if char $\neq 2,3,5$	8 if char $\neq 2,3,5$	0 if char $\neq 2,3,5$
		8 if char $= 2$	12 if char $= 2$	12 if char $= 2$
		6 if char $= 3$	10 if char $= 3$	2 if char $= 3$
		5 if char $= 5$	9 if char $= 5$	1 if char $= 5$

Observation : $\mathrm{HH}^2(A) \hookrightarrow \mathrm{HK}^2(A)$ is not surjective except in type E_8 with char $= 2$.

Dimension of the higher Koszul cohomology I

type	$\mathrm{HK}^0_{hi}(A)$	$\mathrm{HK}^1_{hi}(A)$	$\mathrm{HK}^2_{hi}(A)$
A_n $n \geq 3$	if char $\neq 2$ 0,1 if n even, odd	if char $\neq 2$ 0	n
	if char $= 2$ $m_A + 1$	if char $= 2$ $n - m_A - 1$	
D_n $n \geq 4$	if char $\neq 2$ n, $n-2$ if n even, odd	if char $\neq 2$ 0	if char $\neq 2$ n
	if char $= 2$ $n + m_D$, $n + m_D - 1$ if n even, odd	if char $= 2$ $n - 2$	if char $= 2$ $n + m_D$

Remark. $\mathrm{HK}^\bullet_{hi}(A)$ is not isomorphic to $\mathrm{HK}^\bullet(A)$, except if char $= 2$. Idem in types E_6, E_7 and E_8.

Dimension of the higher Koszul cohomology II

type	$\mathrm{HK}^0_{hi}(A)$	$\mathrm{HK}^1_{hi}(A)$	$\mathrm{HK}^2_{hi}(A)$
E_6	2 if char $\neq 2$	0 if char $\neq 2, 3$	6 if char $\neq 2, 3$
	5 if char $= 2$	4 if char $= 2$	7 if char $= 2$
		1 if char $= 3$	7 if char $= 3$
E_7	7 if char $\neq 2$	0 if char $\neq 2$	7 if char $\neq 2$
	10 if char $= 2$	6 if char $= 2$	10 if char $= 2$
E_8	8 if char $\neq 2$	0 if char $\neq 2, 3$	8 if char $\neq 2$
	12 if char $= 2$	8 if char $= 2$	12 if char $= 2$
		2 if char $= 3$	

From these results, we have proved that the spaces $\mathrm{HK}^0_{hi}(A)$, $\mathrm{HK}^1_{hi}(A)$ and $\mathrm{HK}^2_{hi}(A)$ form a minimal complete list of cohomological invariants for all the ADE preprojective algebras.

The spaces W_0, W_1 and W_2

Let A be the preprojective algebra of type A_3 over a field $\mathbb{F}$, that is, the $\mathbb{F}$-algebra defined by the quiver

$$\overline{Q} \qquad \begin{tikzcd} 0 \arrow[r, bend left, "a_0"] & 1 \arrow[l, bend left, "a_0^*"] \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "a_1^*"] \end{tikzcd}$$

subject to the relations

$$\sigma_0 = -a_0^* a_0, \qquad \sigma_1 = a_0 a_0^* - a_1^* a_1, \qquad \sigma_2 = a_1 a_1^*.$$

The algebra A has dimension 10 and a basis of A over $\mathbb{F}$ is given by the elements e_i for $0 \leqslant i \leqslant 2$, a_i and a_i^* for $0 \leqslant i \leqslant 1$, $a_1 a_0$, $a_1^* a_1$ and $a_0^* a_1^*$. We then define $W_0 = \mathbb{F}\overline{Q}_0 = \mathbb{F}\langle e_0, e_1, e_2 \rangle = k$, $W_1 = \mathbb{F}\overline{Q}_1 = \mathbb{F}\langle a_0, a_1, a_0^*, a_1^* \rangle = V$, $W_2 = \mathbb{F}\langle \sigma_0, \sigma_1, \sigma_2 \rangle = R \subseteq \mathbb{F}\overline{Q}_2$.

Proving that $W_3 = 0$

Now define $W_3 = (V \otimes_k R) \cap (R \otimes_k V)$, viewed inside $\mathbb{F}\overline{Q}_3$. An element u in W_3 can therefore be written as a path in $\mathbb{F}\overline{Q}_3$ in two ways :

$$u = \sum_{i=0}^{1} (\lambda_i a_i \sigma_i + \lambda_i^* a_i^* \sigma_{i+1}) = \sum_{i=0}^{1} (\mu_i \sigma_{i+1} a_i + \mu_i^* \sigma_i a_i^*)$$

with λ_i, λ_i^*, μ_i, μ_i^* in $\mathbb{F}$. Then, in $\mathbb{F}\overline{Q}_3$, we have

$$(-\lambda_0 - \mu_0) a_0 a_0^* a_0 + (-\lambda_1 - \mu_1) a_1 a_1^* a_1 + (\lambda_0^* + \mu_0^*) a_0^* a_0 a_0^*$$
$$+ (\lambda_1^* + \mu_1^*) a_1^* a_1 a_1^* + \lambda_1 a_1 a_0 a_0^* - \lambda_0^* a_0^* a_1^* a_1 + \mu_0 a_1^* a_1 a_0 - \mu_1^* a_0 a_0^* a_1^* = 0$$

so that all the coefficients λ_i, λ_i^*, μ_i and μ_i^* must be zero, hence $u = 0$.

This fact holds for any preprojective algebra not of type A_1 and A_2.

A is not Koszul

Since $W_3 = 0$, the Koszul bimodule complex $K(A)$ is

$$0 \longrightarrow A \otimes_k R \otimes_k A \xrightarrow{d_2} A \otimes_k V \otimes_k A \xrightarrow{d_1} A \otimes_k A \to 0$$

with $d_1(a \otimes_k x \otimes_k a') = ax \otimes_k a' - a \otimes_k xa'$ and

$$d_2(a \otimes_k \sum_{i=1}^{n} \lambda_i x_i y_i \otimes_k a') = \sum_{i=1}^{n} \lambda_i \left(ax_i \otimes_k y_i \otimes_k a' + a \otimes_k x_i \otimes_k y_i a' \right).$$

From $\sigma_1 = a_0 a_0^* - a_1^* a_1$, one has $d_2(a \otimes_k \sigma_1 \otimes_k a') =$
$aa_0 \otimes_k a_0^* \otimes_k a' + a \otimes_k a_0 \otimes_k a_0^* a' - aa_1^* \otimes_k a_1 \otimes_k a' - a \otimes_k a_1^* \otimes_k a_1 a'$
thus $d_2(a_1^* a_1 \otimes \sigma_1 \otimes a_1^* a_1) = 0$. So $\ker(d_2) \neq 0$.
Therefore A is not Koszul.

This fact holds for any ADE preprojective algebra not of type A_1 and A_2. All the non-Dynkin preprojective algebras are Koszul.

Computing the Koszul cohomology I

The Koszul cohomology of A is the homology of the complex $Hom_{A^e}(K(A), A)$, namely the complex

$$0 \to \mathrm{Hom}_{k^e}(k, A) \xrightarrow{b_K^1} \mathrm{Hom}_{k^e}(V, A) \xrightarrow{b_K^2} \mathrm{Hom}_{k^e}(R, A) \to 0.$$

Note that $\mathrm{Hom}_{k^e}(k, A) \cong \bigoplus_{i=0}^{2} e_i A e_i$ has basis $e_0, e_1, e_2, a_1^* a_1$, that f in $\mathrm{Hom}_{k^e}(V, A)$ is defined by $f(a_i) = \lambda_i a_i$ and $f(a_i^*) = \lambda_i^* a_i^*$ for $i = 0, 1$ with λ_i, λ_i^* in $\mathbb{F}$, and that g in $\mathrm{Hom}_{k^e}(R, A)$ is defined by $g(\sigma_i) = \alpha_i e_i$ for $i = 0, 2$ and $g(\sigma_1) = \alpha_1 e_1 + \beta a_1^* a_1$ for some scalars α_i and β.

Then $b_K^1(\sum_{i=0}^{2} u_i e_i + v a_1^* a_1)$ is defined by

$$a_i \mapsto (u_{i+1} - u_i) a_i \text{ and } a_i^* \mapsto (u_i - u_{i+1}) a_i^* \text{ for } i = 0, 1,$$

and $b_K^2(f)$ is defined by

$$\sigma_0 \mapsto 0,\ \sigma_1 \mapsto (\lambda_0 + \lambda_0^* - \lambda_1 - \lambda_1^*) a_1^* a_1 \text{ and } \sigma_2 \mapsto 0.$$

Computing the Koszul cohomology II

It is then easy to see that $\mathrm{HK}^0(A) = \mathbb{F}\langle 1, z_1 = a_1^* a_1 \rangle$, that $\mathrm{HK}^1(A) = \mathbb{F}\langle \bar{\zeta}_0 \rangle$ with $\zeta_0 \in \mathrm{Hom}_{k^e}(V, A)$ defined by $\zeta_0(a_i) = a_i$ and $\zeta_0(a_i^*) = 0$, and that $\mathrm{HK}^2(A) = \mathbb{F}\langle \bar{h}_0, \bar{h}_1, \bar{h}_2 \rangle$ with $h_i \in \mathrm{Hom}_{k^e}(R, A)$ defined by $h_i(\sigma_j) = \delta_{ij} e_i$.

The Koszul cup products can be easily found from the defining formula. It follows that $1 \underset{K}{\smile} x = x \underset{K}{\smile} 1 = x$ for any $x \in \mathrm{HK}^\bullet(A)$ and that all other cup products are 0 in $\mathrm{HK}^\bullet(A)$. For instance, $z_1 \underset{K}{\smile} h_1 : R \to A$ is defined by

$$z_1 \underset{K}{\smile} h_1(\sigma_0) = 0, \qquad z_1 \underset{K}{\smile} h_1(\sigma_1) = z_1, \qquad z_1 \underset{K}{\smile} h_1(\sigma_2) = 0$$

then $z_1 \underset{K}{\smile} h_1 = b_K^2(f)$ where f sends a_0 to a_0 and all other arrows to 0.

Computing the fundamental class

The Koszul homology of A is the homology of the complex $A \otimes_{A^e} K(A)$, namely the complex

$$0 \to A \otimes_{k^e} R \xrightarrow{b_2^K} A \otimes_{k^e} V \xrightarrow{b_1^K} A \otimes_{k^e} k \to 0.$$

From $\sigma_0 = -a_0^* a_0$, one has $b_2^K(a \otimes_{k^e} \sigma_0) = -aa_0^* \otimes_{k^e} a_0 - a_0 a \otimes_{k^e} a_0^*$. In particular, $b_2^K(e_0 \otimes \sigma_0) = -a_0^* \otimes a_0 - a_0 \otimes a_0^*$. Similarly, from $\sigma_1 = a_0 a_0^* - a_1^* a_1$ and $\sigma_2 = a_1 a_1^*$, one has
$b_2^K(e_1 \otimes \sigma_1) = a_0 \otimes a_0^* + a_0^* \otimes a_0 - a_1^* \otimes a_1 - a_1 \otimes a_1^*$,
$b_2^K(e_2 \otimes \sigma_2) = a_1 \otimes a_1^* + a_1^* \otimes a_1$.

So $\omega_0 = \sum_{i=0}^{2} e_i \otimes \sigma_i$ is a Koszul 2-cycle. The non-vanishing class $\overline{\omega}_0 \in \mathrm{HK}_2(A)$ is called the *fundamental class*.

A Poincaré Van den Bergh duality

There is a duality isomorphism $\theta_A : \mathrm{HK}^\bullet(A) \to \mathrm{HK}_{2-\bullet}(A)$ given by $\bar{f} \mapsto \bar{\omega}_0 \underset{K}{\frown} \bar{f}$. Explicitly in our example,

- $\theta_A(1) = \bar{\omega}_0$ and $\theta_A(z_1) = \overline{z_1 \otimes \sigma_1}$ form a basis of $\mathrm{HK}_2(A)$;
- $\theta_A(\bar{\zeta}_0) = \overline{a_0 \otimes a_0^* + a_1 \otimes a_1^*}$ forms a basis of $\mathrm{HK}_1(A)$;
- $\theta_A(\bar{h}_i) = \overline{e_i \otimes e_i}$ for $0 \leqslant i \leqslant 2$ form a basis of $\mathrm{HK}_0(A)$.

Moreover, $\mathrm{HK}^\bullet(A)$ and $\mathrm{HK}_\bullet(A)$ are graded $\mathrm{HK}^\bullet(A)$-bimodules for cup and cap actions respectively. Then θ_A is an isomorphism of graded $\mathrm{HK}^\bullet(A)$-bimodules. Therefore, the cap actions on $\mathrm{HK}_\bullet(A)$ all vanish except $1 \underset{K}{\frown} x = x = x \underset{K}{\frown} 1$ for all $x \in \mathrm{HK}_\bullet(A)$.

This duality isomorphism holds for any preprojective algebra not of type A_1 and A_2, with coefficients in any bimodule M.