Koszul calculus of preprojective algebras

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Plan

Koszul calculus of a quadratic quiver algebra A

Fix a finite quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ and a field $\mathbb{F}$.

The vertex space $k = \mathbb{F}\mathcal{Q}_0$ is a commutative ring by associating with $\mathcal{Q}_0$ a complete set of orthogonal idempotents $e_i,\ i \in \mathcal{Q}_0$.

The arrow space $V = \mathbb{F}\mathcal{Q}_1$ is a k-bimodule. The tensor k-algebra $T_k(V)$ of V is isomorphic to the path algebra $\mathbb{F}\mathcal{Q}$.

Let R be a sub-k-bimodule of $V \otimes_k V \cong \mathbb{F}\mathcal{Q}_2$. The associative k-algebra $A = T_k(V)/(R)$ is called a *quadratic algebra over* $\mathcal{Q}$ *and* $\mathbb{F}$. It is graded by the path length with $A_0 \cong k$ and $A_1 \cong V$.

In the one vertex case $k = \mathbb{F}$, the Koszul calculus was defined and studied in

R. Berger, T. Lambre, A. Solotar, Koszul calculus, *Glasg. Math. J.* 2018.

Defining the Koszul bimodule complex $K(A)$

For any $p \geq 0$, W_p denotes the sub-k-bimodule of $V^{\otimes_k p} \subseteq A^{\otimes_k p}$ defined by

$$W_p = \bigcap_{i+2+j=p} V^{\otimes_k i} \otimes_k R \otimes_k V^{\otimes_k j}.$$

$K(A)$ is the subcomplex of the bar resolution $B(A)$ defined by the sub-A-bimodules $A \otimes_k W_p \otimes_k A$ of $A \otimes_k A^{\otimes_k p} \otimes_k A$.

The differential d of $K(A)$ is given on $A \otimes_k W_p \otimes_k A$ by

$$d(a \otimes_k x_1 \dots x_p \otimes_k a') = ax_1 \otimes_k x_2 \dots x_p \otimes_k a' + (-1)^p a \otimes_k x_1 \dots x_{p-1} \otimes_k x_p a'.$$

When $K(A)$ is a resolution of A, A is said to be Koszul.

Defining the Koszul calculus of A

For any A-bimodule M, we replace the bar resolution $B(A)$ by $K(A)$ in the definition of the Hochschild spaces $\mathrm{HH}_p(A, M)$ and $\mathrm{HH}^p(A, M)$, and we obtain the Koszul spaces $\mathrm{HK}_p(A, M)$ and $\mathrm{HK}^p(A, M)$.

The inclusion $K(A) \hookrightarrow B(A)$ induces linear maps

$$\mathrm{HK}_p(A, M) \to \mathrm{HH}_p(A, M),\ \ \mathrm{HH}^p(A, M) \to \mathrm{HK}^p(A, M)$$

which are always isomorphisms for $p = 0$ and $p = 1$, and if A is Koszul these maps are isomorphisms for any p.

We also need cup and cap products on Koszul cohomology and homology. They are defined by restricting the standard cup and cap products defined on Hochschild cochains and chains. For example the Koszul cup product is defined as follows.

Defining the Koszul calculus of A

Let P, Q be A-bimodules. For Koszul cochains $f \in \mathrm{Hom}_{k^e}(W_p, P)$ and $g \in \mathrm{Hom}_{k^e}(W_q, Q)$, $f \underset{K}{\smile} g \in \mathrm{Hom}_{k^e}(W_{p+q}, P \otimes_A Q)$ is given by

$$(f \underset{K}{\smile} g)(x_1 \dots x_{p+q}) = (-1)^{pq} f(x_1 \dots x_p) \otimes_A g(x_{p+1} \dots x_{p+q}).$$

Then $\tilde{A} = (\mathrm{Hom}_{k^e}(W_\bullet, A), b_K, \underset{K}{\smile})$ is a DG algebra and $\mathrm{Hom}_{k^e}(W_\bullet, M)$ is a DG $\tilde{A}$-bimodule for left and right actions $\underset{K}{\smile}$. Similarly $M \otimes_{k^e} W_\bullet$ is a DG $\tilde{A}$-bimodule for actions $\underset{K}{\frown}$.

The *Koszul calculus of A* consists of the graded associative algebra $(\mathrm{HK}^\bullet(A), \underset{K}{\smile})$ and of the graded $\mathrm{HK}^\bullet(A)$-bimodules $\mathrm{HK}^\bullet(A, M)$ and $\mathrm{HK}_\bullet(A, M)$ for all A-bimodules M. When $M = A$, the Koszul calculus is said to be *restricted*.

Preprojective algebras

Let Q be a quiver whose underlying graph is denoted by Δ. Define a quiver Q^* whose vertex set is Q_0 and whose arrow set is $Q_1^* = \{a^*; a \in Q_1\}$ where $\mathfrak{s}(a^*) = \mathfrak{t}(a)$ and $\mathfrak{t}(a^*) = \mathfrak{s}(a)$.

Let $\overline{Q}$ be the quiver whose vertex set is $\overline{Q}_0 = Q_0$ and whose arrow set is the disjoint union $\overline{Q}_1 = Q_1 \cup Q_1^*$.

Let $\mathbb{F}$ be a field. As before, $k = \mathbb{F}Q_0$, $V = \mathbb{F}\overline{Q}_1$, and $T_k(V) \cong \mathbb{F}\overline{Q}$.

The *preprojective algebra* $A(\Delta)$ is the quadratic k-algebra over $\overline{Q}$ defined by the quadratic relations

$$\sigma_i := \sum_{\substack{a \in Q_1 \\ \mathfrak{t}(a)=i}} aa^* - \sum_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} a^* a$$

for all $i \in Q_0$.

Why preprojective algebras are of interest for us ?

If a quadratic quiver algebra A is Koszul, its Koszul invariants are the same as its Hochschild invariants. If A is not Koszul, new invariants can be expected. The following is standard.

Proposition

Assume that the graph Δ is distinct from A_1 and A_2. The following are equivalent.
(i) Δ is Dynkin of type ADE.
(ii) $A(\Delta)$ is not Koszul.
(iii) $A(\Delta)$ is finite dimensional.

In our paper, we compute the restricted Koszul calculus of the ADE preprojective algebras A. From these computations, we prove that the inclusion $\mathrm{HH}^2(A) \hookrightarrow \mathrm{HK}^2(A)$ is not surjective, except in type E_8 with $\mathrm{char}(\mathbb{F}) = 2$.

A Poincaré Van den Bergh duality

In our paper, we prove the following.

Theorem

Let A be the preprojective algebra of a connected graph Δ distinct from A_1 and A_2, over a field $\mathbb{F}$. Let M be an A-bimodule.
(i) The complex $K(A)$ has length 2. In particular, $\mathrm{HK}^p(A, M) = \mathrm{HK}_p(A, M) = 0$ *for any* $p > 2$.
(ii) The $\mathrm{HK}^\bullet(A)$*-bimodules* $\mathrm{HK}^\bullet(A, M)$ *and* $\mathrm{HK}_{2-\bullet}(A, M)$ *are isomorphic.*

Consequently, the Koszul calculus in homology can be deduced from the Koszul calculus in cohomology.
Now we want to explain (i) and (ii) in more details.

(i) $K(A)$ has length 2

Since $W_p = (W_{p-1} \otimes_k V) \cap (V \otimes_k W_{p-1})$ for all $p > 2$ and $W_2 := R \neq 0$, it suffices to prove that

$$W_3 := (R \otimes_k V) \cap (V \otimes_k R) = 0.$$

We assume that $\Delta \neq \mathrm{A}_1$ and prove that $W_3 \neq 0$ implies $\Delta = \mathrm{A}_2$:

Let u be a non-zero element in W_3. We may assume that u is in eW_3f for some vertices e and f. Then u can be written uniquely in the basis of $R \otimes_k V$ and in the basis of $V \otimes_k R$. Comparing the two decompositions, we show that

$$e \neq f,\ Q_0 = \{e, f\},\ |Q_1| = 1.$$

Thus $\Delta =$A$_2$.

(ii) the duality $\mathrm{HK}^p(A, M) \cong \mathrm{HK}_{2-p}(A, M)$

This isomorphism comes from an explicit isomorphism from the complex defining $\mathrm{HK}^p(A, M)$ to the complex defining $\mathrm{HK}_{2-p}(A, M)$:

Theorem

Define $\omega_0 = \sum_i e_i \otimes \sigma_i$. It is a Koszul 2-cycle with coefficients in A. For each Koszul p-cochain f with coefficients in M, we define the Koszul $(2-p)$-chain $\theta_M(f)$ with coefficients in M by

$$\theta_M(f) = \omega_0 \underset{K}{\frown} f.$$

Then θ_M is an isomorphism of complexes. Moreover, the equalities

$$\theta_{M \otimes_A N}(f \underset{K}{\smile} g) = \theta_M(f) \underset{K}{\frown} g = f \underset{K}{\frown} \theta_N(g)$$

hold for Koszul cochains f and g with coefficients in M and N.

A fundamental class for the duality

Passing the isomorphism of complexes

$$\theta_M = \omega_0 \underset{K}{\frown} - : \mathrm{Hom}_{k^e}(W_\bullet, M) \to M \otimes_{k^e} W_{2-\bullet}$$

to homology, we get a linear isomorphism

$$H(\theta_M) = \overline{\omega}_0 \underset{K}{\frown} - : \mathrm{HK}^\bullet(A, M) \to \mathrm{HK}_{2-\bullet}(A, M).$$

So the class $\overline{\omega}_0 \in \mathrm{HK}_2(A)$ is an analogue of the *fundamental class* used in the classical Poincaré's duality for singular (co)homology.

A generalisation of the 2-Calabi-Yau property

Take $M = A^e := A \otimes A^{op}$ in our duality and identify $A^e \otimes_{k^e} W_\bullet$ to $K(A)$. We obtain an isomorphism of complexes

$$\theta_{A^e} : \mathrm{Hom}_{k^e}(W_\bullet, A^e) \to K(A)[-2] \text{ in } A\text{-Bimod}$$

inducing an isomorphism

$$\mathrm{RHom}_{A^e}(K(A), A^e) \to K(A)[-2] \text{ in } \mathcal{D}^b(A\text{-Bimod}).$$

Then we say that A is *Koszul complex Calabi-Yau of dimension 2*.

Assume that Δ is not Dynkin ADE, so that A is Koszul. Then $K(A) \cong A$ in $\mathcal{D}^b(A\text{-Bimod})$, implying an isomorphism

$$\mathrm{RHom}_{A^e}(A, A^e) \to A[-2] \text{ in } \mathcal{D}^b(A\text{-Bimod}).$$

We recover that A is 2-Calabi-Yau in Ginzburg's sense.

The situation is new in the ADE types !

Passing to homology in 2-Kc-Calabi-Yau algebras, we get
(i) The A-bimodule $\mathrm{HK}^2(A, A^e)$ is isomorphic to the A-bimodule A.
(ii) $\mathrm{HK}^1(A, A^e) = 0$.
(iii) The A-bimodule $\mathrm{HK}^0(A, A^e)$ is isomorphic to the A-bimodule $H_2(K(A))$.

Passing to homology in 2-Calabi-Yau algebras, we get
(i) The A-bimodule $\mathrm{HH}^2(A, A^e)$ is isomorphic to the A-bimodule A.
(ii) $\mathrm{HH}^p(A, A^e) \cong 0$ for $p \neq 2$.

Assume that A is Dynkin ADE. Then $H_2(K(A)) \neq 0$ because A is not Koszul. Therefore, since $\mathrm{HH}^0(A, A^e) \cong \mathrm{HK}^0(A, A^e)$, A is not 2-Calabi-Yau in Ginzburg's sense. However, A is 2-Kc-Calabi-Yau !

An n-generalisation

Definition

Let $A = T_k(V)/(R)$ be a quadratic k-algebra over a finite quiver $\mathcal{Q}$. Let $n \geq 0$ be an integer. We say that A is Koszul complex Calabi-Yau of dimension n (n-Kc-Calabi-Yau) if
(i) the Koszul bimodule complex $K(A)$ of A has length n, and
(ii) $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in $\mathcal{D}^b(A\text{-Bimod})$.

Assume that A is Koszul. Then A is n-Kc-Calabi-Yau if and only if A is n-Calabi-Yau.

Conjecture

Let $A = T_k(V)/(R)$ be a quadratic k-algebra over a finite quiver $\mathcal{Q}$. If A is n-Calabi-Yau and n-Kc-Calabi-Yau, then A is Koszul (proved if $n \leq 3$).

Duality for n-Kc-Calabi-Yau algebras

Theorem

Let A be an n-Kc-Calabi-Yau algebra. Then for any A-bimodule M, the vector spaces $\mathrm{HK}^p(A, M)$ and $\mathrm{HK}_{n-p}(A, M)$ are isomorphic.

Sketch of proof : For any finitely generated projective A-bimodule P, we have a standard isomorphism

$$M \otimes_{A^e} \mathrm{Hom}_{A^e}(P, A^e) \cong \mathrm{Hom}_{A^e}(P, M) \text{ in } \mathsf{Vect}_{\mathbb{F}} \ \text{ inducing}$$

$$M \overset{L}{\otimes}_{A^e} \mathrm{RHom}_{A^e}(K(A), A^e) \cong \mathrm{RHom}_{A^e}(K(A), M) \text{ in } \mathcal{D}^b(\mathsf{Vect}_{\mathbb{F}}).$$

Combining with $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ in $\mathcal{D}^b(A\text{-Bimod})$, we conclude

$$\mathrm{RHom}_{A^e}(K(A), M) \cong M \overset{L}{\otimes}_{A^e} K(A)[-n] \text{ in } \mathcal{D}^b(\mathsf{Vect}_{\mathbb{F}}).$$

Towards an enriched duality isomorphism

Definition

Let A be an n-Kc-Calabi-Yau algebra. The image $c \in \mathrm{HK}_n(A)$ of the unit 1 of A by the duality isomorphism $\mathrm{HK}^0(A) \cong \mathrm{HK}_n(A)$ is called the fundamental class of the n-Kc-Calabi-Yau algebra A.

We want to define strong Kc-Calabi-Yau algebras.

For this we need to work with DG $\tilde{A}$-bimodules in A-Bimod, where $\tilde{A}$ denotes the DG algebra $\mathrm{Hom}_{A^e}(K(A), A)$.

Following Yekutieli's book (Derived categories. Cambridge Studies in Advanced Mathematics 183, CUP, 2020), $\mathcal{C}(\tilde{A}, A\text{-Bimod})$ ($\mathcal{C}(\tilde{A}, \mathrm{Vect}_{\mathbb{F}})$) denotes the category of DG $\tilde{A}$-bimodules in A-Bimod (in $\mathrm{Vect}_{\mathbb{F}}$), and the associated bounded derived categories are defined.

Duality for strong generalised Calabi-Yau algebras

Definition

Let A be an n-Kc-Calabi-Yau algebra. Then A is said to be strong n-Kc-Calabi-Yau if the derived functor of the endofunctor $\mathrm{Hom}_{A^e}(-, A^e)$ *of* $\mathcal{C}^b(\tilde{A}, A\text{-Bimod})$ *exists and if* $\mathrm{RHom}_{A^e}(K(A), A^e) \cong K(A)[-n]$ *in* $\mathcal{D}^b(\tilde{A}, A\text{-Bimod})$.

Theorem

Let A be a strong n-Kc-Calabi-Yau algebra with fundamental class c. We assume that the derived functors of the functors $\mathrm{Hom}_{A^e}(-, A)$ *and* $A \otimes_{A^e} -$ *from* $\mathcal{C}^b(\tilde{A}, A\text{-Bimod})$ *to* $\mathcal{C}^b(\tilde{A}, \mathrm{Vect}_{\mathbb{F}})$ *exist.*
Then

$$c \underset{K}{\frown} - : \mathrm{HK}^\bullet(A) \to \mathrm{HK}_{n-\bullet}(A)$$

is an isomorphism of graded $\mathrm{HK}^\bullet(A)$*-bimodules.*

Hochschild homology and preprojective algebras

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