Koszul calculus for N-homogeneous algebras

ROLAND BERGER

Université de Saint-Etienne
Institut Camille Jordan
UMR 5208

Koszul calculus for N-homogeneous algebras

A is any N-homogeneous algebra

Fix a vector space V over a field k and an integer $N \geq 2$. Fix a subspace R of $V^{\otimes N}$.

A is any N-homogeneous algebra

Fix a vector space V over a field k and an integer $N \geq 2$. Fix a subspace R of $V^{\otimes N}$.

The graded associative algebra $A = T(V)/(R)$ is called an *N-homogeneous algebra*. If $N = 2$, A is a quadratic algebra.

A is any N-homogeneous algebra

Fix a vector space V over a field k and an integer $N \geq 2$. Fix a subspace R of $V^{\otimes N}$.

The graded associative algebra $A = T(V)/(R)$ is called an *N-homogeneous algebra*. If $N = 2$, A is a quadratic algebra.

Koszul calculus for quadratic algebras was defined and studied in

R. Berger, T. Lambre, A. Solotar, Koszul calculus, *Glasg. Math. J.* 2018.

A is any N-homogeneous algebra

Fix a vector space V over a field k and an integer $N \geq 2$. Fix a subspace R of $V^{\otimes N}$.

The graded associative algebra $A = T(V)/(R)$ is called an *N-homogeneous algebra*. If $N = 2$, A is a quadratic algebra.

Koszul calculus for quadratic algebras was defined and studied in

R. Berger, T. Lambre, A. Solotar, Koszul calculus, *Glasg. Math. J.* 2018.

Aim : construct a Koszul calculus for the N-homogeneous algebra A.

A is any N-homogeneous algebra

Fix a vector space V over a field k and an integer $N \geq 2$. Fix a subspace R of $V^{\otimes N}$.

The graded associative algebra $A = T(V)/(R)$ is called an *N-homogeneous algebra*. If $N = 2$, A is a quadratic algebra.

Koszul calculus for quadratic algebras was defined and studied in

R. Berger, T. Lambre, A. Solotar, Koszul calculus, *Glasg. Math. J.* 2018.

Aim : construct a Koszul calculus for the N-homogeneous algebra A.

R. Berger, Koszul calculus for N-homogeneous algebras, *J. Algebra* 2019.

Constructing the bimodule Koszul complex of A

For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by

$$W_p = \bigcap_{i+N+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

Constructing the bimodule Koszul complex of A

For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by

$$W_p = \bigcap_{i+N+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

When $N = 2$ and $A = S(V)$, $W_p \cong \bigwedge^p V$.

Constructing the bimodule Koszul complex of A

For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by

$$W_p = \bigcap_{i+N+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

When $N = 2$ and $A = S(V)$, $W_p \cong \bigwedge^p V$.

For any N, one has $W_p = V^{\otimes p}$ if $0 \leq p < N$ and $W_N = R$.

Constructing the bimodule Koszul complex of A

For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by

$$W_p = \bigcap_{i+N+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

When $N = 2$ and $A = S(V)$, $W_p \cong \bigwedge^p V$.

For any N, one has $W_p = V^{\otimes p}$ if $0 \leq p < N$ and $W_N = R$.

Only $W_0 = k$, $W_1 = V$, $W_N = R$, $W_{N+1} = (R \otimes V) \cap (V \otimes R)$,... are of interest in the construction. We set

Constructing the bimodule Koszul complex of A

For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by

$$W_p = \bigcap_{i+N+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

When $N = 2$ and $A = S(V)$, $W_p \cong \bigwedge^p V$.

For any N, one has $W_p = V^{\otimes p}$ if $0 \leq p < N$ and $W_N = R$.

Only $W_0 = k$, $W_1 = V$, $W_N = R$, $W_{N+1} = (R \otimes V) \cap (V \otimes R)$,... are of interest in the construction. We set

$\nu(2p') = p'N$ and $\nu(2p'+1) = p'N + 1$.

The bimodule Koszul complex $K(A)$

$d : A \otimes W_{\nu(p)} \otimes A \to A \otimes W_{\nu(p-1)} \otimes A$ is defined by

The bimodule Koszul complex $K(A)$

$d : A \otimes W_{\nu(p)} \otimes A \to A \otimes W_{\nu(p-1)} \otimes A$ is defined by

If $p = 2p' + 1$, then

$$d(a \otimes x_1 \dots x_{p'N+1} \otimes a') = ax_1 \otimes x_2 \dots x_{p'N+1} \otimes a' - a \otimes x_1 \dots x_{p'N} \otimes x_{p'N+1} a'.$$

The bimodule Koszul complex $K(A)$

$d : A \otimes W_{\nu(p)} \otimes A \rightarrow A \otimes W_{\nu(p-1)} \otimes A$ is defined by

If $p = 2p' + 1$, then

$$d(a \otimes x_1 \dots x_{p'N+1} \otimes a') = ax_1 \otimes x_2 \dots x_{p'N+1} \otimes a' - a \otimes x_1 \dots x_{p'N} \otimes x_{p'N+1} a'.$$

If $p = 2p'$, then

$$d(a \otimes x_1 \dots x_{p'N} \otimes a') =$$

$$\sum_{i=0}^{i=N-1} ax_1 \dots x_i \otimes x_{i+1} \dots x_{i+p'N-N+1} \otimes x_{i+p'N-N+2} \dots x_{p'N} a'.$$

Defining $HK_p(A, M)$ for any A-bimodule M

The complex $(M \otimes W_{\nu(\bullet)}, b_K)$ is defined as follows :

$$b_K : M \otimes W_{\nu(p)} \to M \otimes W_{\nu(p-1)}$$

Defining $HK_p(A, M)$ for any A-bimodule M

The complex $(M \otimes W_{\nu(\bullet)}, b_K)$ is defined as follows :

$$b_K : M \otimes W_{\nu(p)} \to M \otimes W_{\nu(p-1)}$$

if $p = 2p' + 1$,

$$b_K(m \otimes x_1 \dots x_{p'N+1}) = mx_1 \otimes x_2 \dots x_{p'N+1} - x_{p'N+1}m \otimes x_1 \dots x_{p'N},$$

Defining $HK_p(A, M)$ for any A-bimodule M

The complex $(M \otimes W_{\nu(\bullet)}, b_K)$ is defined as follows :

$$b_K : M \otimes W_{\nu(p)} \to M \otimes W_{\nu(p-1)}$$

if $p = 2p' + 1$,

$$b_K(m \otimes x_1 \ldots x_{p'N+1}) = mx_1 \otimes x_2 \ldots x_{p'N+1} - x_{p'N+1}m \otimes x_1 \ldots x_{p'N},$$

if $p = 2p'$,

$$b_k(m \otimes x_1 \ldots x_{p'N}) =$$

$$\sum_{i=0}^{i=N-1} x_{i+p'N-N+2} \ldots x_{p'N} m x_1 \ldots x_i \otimes x_{i+1} \ldots x_{i+p'N-N+1}.$$

Defining $HK^p(A, M)$

The complex $(Hom_k(W_{\nu(\bullet)}, M), b_K)$ is defined as follows :

$$b_K : Hom_k(W_{\nu(p)}, M) \to Hom_k(W_{\nu(p+1)}, M)$$

Defining $HK^p(A, M)$

The complex $(Hom_k(W_{\nu(\bullet)}, M), b_K)$ is defined as follows :

$$b_K : Hom_k(W_{\nu(p)}, M) \to Hom_k(W_{\nu(p+1)}, M)$$

if $p = 2p'$,

$$b_K(f)(x_1 \dots x_{p'N+1}) = f(x_1 \dots x_{p'N})x_{p'N+1} - x_1 f(x_2 \dots x_{p'N+1}),$$

Defining $HK^p(A, M)$

The complex $(Hom_k(W_{\nu(\bullet)}, M), b_K)$ is defined as follows :

$$b_K : Hom_k(W_{\nu(p)}, M) \to Hom_k(W_{\nu(p+1)}, M)$$

if $p = 2p'$,

$$b_K(f)(x_1 \dots x_{p'N+1}) = f(x_1 \dots x_{p'N})x_{p'N+1} - x_1 f(x_2 \dots x_{p'N+1}),$$

if $p = 2p' + 1$,

$b_k(f)(x_1 \dots x_{p'N+N}) =$

$$\sum_{i=0}^{i=N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+p'N+1}) x_{i+p'N+2} \dots x_{p'N+N}.$$

Koszul calculus for N-homogeneous algebras

Definition of the Koszul cup product

For $f : W_{\nu(p)} \to P$ and $g : W_{\nu(q)} \to Q$, define

$$f \underset{K}{\smile} g : W_{\nu(p+q)} \to P \otimes_A Q$$

Definition of the Koszul cup product

For $f : W_{\nu(p)} \to P$ and $g : W_{\nu(q)} \to Q$, define

$$f \underset{K}{\smile} g : W_{\nu(p+q)} \to P \otimes_A Q$$

if p or q is even, then $\nu(p+q) = \nu(p) + \nu(q)$ and

$$(f \underset{K}{\smile} g)(x_1 \ldots x_{\nu(p+q)}) = f(x_1 \ldots x_{\nu(p)}) \otimes_A g(x_{\nu(p)+1} \ldots x_{\nu(p+q)}),$$

Definition of the Koszul cup product

For $f : W_{\nu(p)} \to P$ and $g : W_{\nu(q)} \to Q$, define

$$f \underset{K}{\smile} g : W_{\nu(p+q)} \to P \otimes_A Q$$

if p or q is even, then $\nu(p+q) = \nu(p) + \nu(q)$ and

$$(f \underset{K}{\smile} g)(x_1 \ldots x_{\nu(p+q)}) = f(x_1 \ldots x_{\nu(p)}) \otimes_A g(x_{\nu(p)+1} \ldots x_{\nu(p+q)}),$$

if $p = 2p' + 1$ and $q = 2q' + 1$, then $\nu(p) = p'N + 1$, $\nu(q) = q'N + 1$, $\nu(p+q) = p'N + q'N + N = \nu(p) + \nu(q) + N - 2$ and

$$(f \underset{K}{\smile} g)(x_1 \ldots x_{p'N+q'N+N}) =$$

$$- \sum_{0 \leq i+j \leq N-2} x_1 \ldots x_i f(x_{i+1} \ldots x_{i+p'N+1}) x_{i+p'N+2} \ldots x_{p'N+N-j-1}$$

$$\otimes_A g(x_{p'N+N-j} \ldots x_{p'N+q'N+N-j}) x_{p'N+q'N+N-j+1} \ldots x_{p'N+q'N+N}.$$

Definition of the Koszul cap products

For $f : W_{\nu(p)} \to P$ and $z = m \otimes x_1 \dots x_{\nu(q)} \in M \otimes W_{\nu(q)}$ with $q \geq p$, define

$$f \underset{K}{\frown} z \in (P \otimes_A M) \otimes W_{\nu(q-p)}$$

$$z \underset{K}{\frown} f \in (M \otimes_A P) \otimes W_{\nu(q-p)}$$

as follows.

Definition of the Koszul cap products

For $f : W_{\nu(p)} \to P$ and $z = m \otimes x_1 \dots x_{\nu(q)} \in M \otimes W_{\nu(q)}$ with $q \geq p$, define

$$f \underset{K}{\frown} z \in (P \otimes_A M) \otimes W_{\nu(q-p)}$$

$$z \underset{K}{\frown} f \in (M \otimes_A P) \otimes W_{\nu(q-p)}$$

as follows.

If p even or q odd, then $\nu(q) = \nu(q-p) + \nu(p)$ and

$$f \underset{K}{\frown} z = (f(x_{\nu(q-p)+1} \dots x_{\nu(q)}) \otimes_A m) \otimes x_1 \dots x_{\nu(q-p)}$$

Definition of the Koszul cap products

For $f : W_{\nu(p)} \to P$ and $z = m \otimes x_1 \dots x_{\nu(q)} \in M \otimes W_{\nu(q)}$ with $q \geq p$, define

$$f \underset{K}{\frown} z \in (P \otimes_A M) \otimes W_{\nu(q-p)}$$

$$z \underset{K}{\frown} f \in (M \otimes_A P) \otimes W_{\nu(q-p)}$$

as follows.

If p even or q odd, then $\nu(q) = \nu(q-p) + \nu(p)$ and

$$f \underset{K}{\frown} z = (f(x_{\nu(q-p)+1} \dots x_{\nu(q)}) \otimes_A m) \otimes x_1 \dots x_{\nu(q-p)}$$

$$z \underset{K}{\frown} f = (-1)^{pq}(m \otimes_A f(x_1 \dots x_{\nu(p)})) \otimes x_{\nu(p)+1} \dots x_{\nu(q)}$$

Definition of the Koszul cap products

If $p = 2p' + 1$ and $q = 2q'$, then $\nu(q) = \nu(q-p) + \nu(p) - N + 2$ and

$(f \underset{K}{\frown} z) = -\sum_{0 \leq i+j \leq N-2}$

$$(x_{q'N-p'N-N+i+2} \ldots x_{q'N-p'N-j-1} f(x_{q'N-p'N-j} \ldots x_{q'N-j})$$

$$\otimes_A x_{q'N-j+1} \ldots x_{q'N}\, m\, x_1 \ldots x_i) \otimes x_{i+1} \ldots x_{i+q'N-p'N-N+1}$$

Definition of the Koszul cap products

If $p = 2p' + 1$ and $q = 2q'$, then $\nu(q) = \nu(q-p) + \nu(p) - N + 2$ and

$(f \underset{K}{\frown} z) = -\sum_{0 \le i+j \le N-2}$

$$(x_{q'N-p'N-N+i+2} \dots x_{q'N-p'N-j-1} f(x_{q'N-p'N-j} \dots x_{q'N-j})$$

$$\otimes_A x_{q'N-j+1} \dots x_{q'N}\, m\, x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+q'N-p'N-N+1}$$

$(z \underset{K}{\frown} f) = \sum_{0 \le i+j \le N-2}$

$$(x_{q'N-j+1} \dots x_{q'N}\, m\, x_1 \dots x_i \otimes_A f(x_{i+1} \dots x_{i+p'N+1})$$

$$x_{i+p'N+2} \dots x_{p'N+N-j-1}) \otimes x_{p'N+N-j} \dots x_{q'N-j}.$$

Koszul calculus for N-homogeneous algebras

Everything goes well !

Let $A = T(V)/(R)$ be an N-homogeneous algebra. Let P and Q be A-bimodules.

Everything goes well !

Let $A = T(V)/(R)$ be an N-homogeneous algebra. Let P and Q be A-bimodules.

For any Koszul p-cochain f with coefficients in P and Koszul q-cochain g with coefficients in Q, one has

Everything goes well !

Let $A = T(V)/(R)$ be an N-homogeneous algebra. Let P and Q be A-bimodules.

For any Koszul p-cochain f with coefficients in P and Koszul q-cochain g with coefficients in Q, one has

$$b_K(f \underset{K}{\smile} g) = b_K(f) \underset{K}{\smile} g + (-1)^p f \underset{K}{\smile} b_K(g).$$

Everything goes well !

Let $A = T(V)/(R)$ be an N-homogeneous algebra. Let P and Q be A-bimodules.

For any Koszul p-cochain f with coefficients in P and Koszul q-cochain g with coefficients in Q, one has

$$b_K(f \underset{K}{\smile} g) = b_K(f) \underset{K}{\smile} g + (-1)^p f \underset{K}{\smile} b_K(g).$$

Thus we obtain a Koszul cup product on Koszul cohomology classes

Everything goes well !

Let $A = T(V)/(R)$ be an N-homogeneous algebra. Let P and Q be A-bimodules.

For any Koszul p-cochain f with coefficients in P and Koszul q-cochain g with coefficients in Q, one has

$$b_K(f \underset{K}{\smile} g) = b_K(f) \underset{K}{\smile} g + (-1)^p f \underset{K}{\smile} b_K(g).$$

Thus we obtain a Koszul cup product on Koszul cohomology classes

$$\underset{K}{\smile} : HK^p(A, P) \otimes HK^q(A, Q) \to HK^{p+q}(A, P \otimes_A Q).$$

Everything goes well !

Let $A = T(V)/(R)$ be an N-homogeneous algebra. Let P and Q be A-bimodules.

For any Koszul p-cochain f with coefficients in P and Koszul q-cochain g with coefficients in Q, one has

$$b_K(f \underset{K}{\smile} g) = b_K(f) \underset{K}{\smile} g + (-1)^p f \underset{K}{\smile} b_K(g).$$

Thus we obtain a Koszul cup product on Koszul cohomology classes

$$\underset{K}{\smile} : HK^p(A, P) \otimes HK^q(A, Q) \to HK^{p+q}(A, P \otimes_A Q).$$

Claim : this product is associative. One can assume $N > 2$.

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

$$as(f,g,h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

$$as(f,g,h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

1) If $\nu(p+q+r) = \nu(p) + \nu(q) + \nu(r)$, then $as(f,g,h) = 0$.

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

$$as(f,g,h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

1) If $\nu(p+q+r) = \nu(p) + \nu(q) + \nu(r)$, then $as(f,g,h) = 0$.

2) $p = 2p'$, $q = 2q' + 1$, $r = 2r' + 1$: $as(f,g,h) = 0$ if f is a cocycle.

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

$$as(f,g,h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

1) If $\nu(p+q+r) = \nu(p) + \nu(q) + \nu(r)$, then $as(f,g,h) = 0$.

2) $p = 2p'$, $q = 2q'+1$, $r = 2r'+1$: $as(f,g,h) = 0$ if f is a cocycle.

3) $p = 2p'+1$, $q = 2q'$, $r = 2r'+1$: $as(f,g,h) = 0$ if g is a cocycle.

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

$$as(f,g,h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

1) If $\nu(p+q+r) = \nu(p) + \nu(q) + \nu(r)$, then $as(f,g,h) = 0$.

2) $p = 2p'$, $q = 2q'+1$, $r = 2r'+1$: $as(f,g,h) = 0$ if f is a cocycle.

3) $p = 2p'+1$, $q = 2q'$, $r = 2r'+1$: $as(f,g,h) = 0$ if g is a cocycle.

4) $p = 2p'+1$, $q = 2q'+1$, $r = 2r'$: $as(f,g,h) = 0$ if h is a cocycle.

Associativity on classes : a scheme of proof

For Koszul cochains f, g and h, we define their associator

$$as(f,g,h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

1) If $\nu(p+q+r) = \nu(p) + \nu(q) + \nu(r)$, then $as(f,g,h) = 0$.

2) $p = 2p'$, $q = 2q' + 1$, $r = 2r' + 1$: $as(f,g,h) = 0$ if f is a cocycle.

3) $p = 2p' + 1$, $q = 2q'$, $r = 2r' + 1$: $as(f,g,h) = 0$ if g is a cocycle.

4) $p = 2p' + 1$, $q = 2q' + 1$, $r = 2r'$: $as(f,g,h) = 0$ if h is a cocycle.

5) $p = 2p' + 1$, $q = 2q' + 1$, $r = 2r' + 1$: $as(f,g,h) = 0$ is always a coboundary.

A non-associative example at the cochain level

A is the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by complex parameters a, b and c.

A non-associative example at the cochain level

A is the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by complex parameters a, b and c.

A is defined by two generators x and y and two cubic relations $r_1 = 0$ and $r_2 = 0$ where

A non-associative example at the cochain level

A is the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by complex parameters a, b and c.

A is defined by two generators x and y and two cubic relations $r_1 = 0$ and $r_2 = 0$ where

$$r_1 = ay^2x + byxy + axy^2 + cx^3, \; r_2 = ax^2y + bxyx + ayx^2 + cy^3.$$

A non-associative example at the cochain level

A is the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by complex parameters a, b and c.

A is defined by two generators x and y and two cubic relations $r_1 = 0$ and $r_2 = 0$ where

$$r_1 = ay^2x + byxy + axy^2 + cx^3, \; r_2 = ax^2y + bxyx + ayx^2 + cy^3.$$

We know that $W_4 = \mathbb{C}w$, where $w = xr_1 + yr_2$.

A non-associative example at the cochain level

A is the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by complex parameters a, b and c.

A is defined by two generators x and y and two cubic relations $r_1 = 0$ and $r_2 = 0$ where

$$r_1 = ay^2x + byxy + axy^2 + cx^3, \; r_2 = ax^2y + bxyx + ayx^2 + cy^3.$$

We know that $W_4 = \mathbb{C}w$, where $w = xr_1 + yr_2$.

$f, g, h : V \to A$ are the Koszul 1-cochains defined by $f(x) = g(x) = x$ and $h(x) = 1$ for $x \in V$. We are in Case 5 of the previous proof.

A non-associative example at the cochain level

A is the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by complex parameters a, b and c.

A is defined by two generators x and y and two cubic relations $r_1 = 0$ and $r_2 = 0$ where

$$r_1 = ay^2x + byxy + axy^2 + cx^3, \; r_2 = ax^2y + bxyx + ayx^2 + cy^3.$$

We know that $W_4 = \mathbb{C}w$, where $w = xr_1 + yr_2$.

$f, g, h : V \to A$ are the Koszul 1-cochains defined by $f(x) = g(x) = x$ and $h(x) = 1$ for $x \in V$. We are in Case 5 of the previous proof.

Then $as(f, g, h)(w) = (a - b)(xy - yx)(x - y)$ is not zero in A.

Koszul calculus for N-homogeneous algebras

The fundamental 1-class

The Koszul 1-cocycles $f : V \to M$ are linear maps satisfying

The fundamental 1-class

The Koszul 1-cocycles $f : V \to M$ are linear maps satisfying

$$\sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1}) x_{i+2} \dots x_N = 0$$

for any $x_1 \dots x_N$ in R.

The fundamental 1-class

The Koszul 1-cocycles $f : V \to M$ are linear maps satisfying

$$\sum_{0 \le i \le N-1} x_1 \dots x_i f(x_{i+1}) x_{i+2} \dots x_N = 0$$

for any $x_1 \dots x_N$ in R.

Then f extends to a derivation $D_f : A \to M$. And conversely.

The fundamental 1-class

The Koszul 1-cocycles $f : V \to M$ are linear maps satisfying

$$\sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1}) x_{i+2} \dots x_N = 0$$

for any $x_1 \dots x_N$ in R.

Then f extends to a derivation $D_f : A \to M$. And conversely.

The Euler derivation $D : A \to A$ defined by $D(a) = pa$ if $a \in A_p$ restricts to a Koszul 1-cocycle $e_A : V \to A, \quad x \mapsto x$.

The fundamental 1-class

The Koszul 1-cocycles $f : V \to M$ are linear maps satisfying

$$\sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1}) x_{i+2} \dots x_N = 0$$

for any $x_1 \dots x_N$ in R.

Then f extends to a derivation $D_f : A \to M$. And conversely.

The Euler derivation $D : A \to A$ defined by $D(a) = pa$ if $a \in A_p$ restricts to a Koszul 1-cocycle $e_A : V \to A, \;\; x \mapsto x$.

If $V \neq 0$, the class $\overline{e}_A$ is not zero and is called the *fundamental 1-class* of A.

The fundamental formulas

For a p-cochain $f : W_{\nu(p)} \to P$ and a q-cochain $g : W_{\nu(q)} \to Q$, we define their Koszul cup bracket when P or Q is equal to A, by

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f.$$

The fundamental formulas

For a p-cochain $f : W_{\nu(p)} \to P$ and a q-cochain $g : W_{\nu(q)} \to Q$, we define their Koszul cup bracket when P or Q is equal to A, by

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f.$$

Then for any Koszul p-cochain $f : W_{\nu(p)} \to M$, we have

The fundamental formulas

For a p-cochain $f : W_{\nu(p)} \to P$ and a q-cochain $g : W_{\nu(q)} \to Q$, we define their Koszul cup bracket when P or Q is equal to A, by

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f.$$

Then for any Koszul p-cochain $f : W_{\nu(p)} \to M$, we have

$[e_A, f]_{\underset{K}{\smile}} = -\, b_K(f)$ if p is even,

$[e_A, f]_{\underset{K}{\smile}} = (1 - N)\, b_K(f)$ if p is odd.

The fundamental formulas

For a p-cochain $f : W_{\nu(p)} \to P$ and a q-cochain $g : W_{\nu(q)} \to Q$, we define their Koszul cup bracket when P or Q is equal to A, by

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f.$$

Then for any Koszul p-cochain $f : W_{\nu(p)} \to M$, we have

$[e_A, f]_{\underset{K}{\smile}} = -\, b_K(f)$ if p is even,

$[e_A, f]_{\underset{K}{\smile}} = (1 - N)\, b_K(f)$ if p is odd.

Similarly for any Koszul q-chain $z \in M \otimes W_{\nu(q)}$, we have

The fundamental formulas

For a p-cochain $f : W_{\nu(p)} \to P$ and a q-cochain $g : W_{\nu(q)} \to Q$, we define their Koszul cup bracket when P or Q is equal to A, by

$$[f, g]_{\underset{K}{\smile}} = f \underset{K}{\smile} g - (-1)^{pq} g \underset{K}{\smile} f.$$

Then for any Koszul p-cochain $f : W_{\nu(p)} \to M$, we have

$$[e_A, f]_{\underset{K}{\smile}} = -\, b_K(f) \text{ if } p \text{ is even,}$$

$$[e_A, f]_{\underset{K}{\smile}} = (1 - N)\, b_K(f) \text{ if } p \text{ is odd.}$$

Similarly for any Koszul q-chain $z \in M \otimes W_{\nu(q)}$, we have

$$[e_A, z]_{\underset{K}{\frown}} = -\, b_K(z) \text{ if } q \text{ is odd,}$$

$$[e_A, z]_{\underset{K}{\frown}} = (1 - N)\, b_K(z) \text{ if } q \text{ is even.}$$

Koszul calculus for N-homogeneous algebras

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

$$\overline{e}_A \underset{K}{\smile} - : HK^p(A, M) \to HK^{p+1}(A, M)$$

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

$$\overline{e}_A \underset{K}{\smile} - : HK^p(A, M) \to HK^{p+1}(A, M)$$

$$\overline{e}_A \underset{K}{\frown} - : HK_q(A, M) \to HK_{q-1}(A, M)$$

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

$$\overline{e}_A \underset{K}{\smile} - : HK^p(A, M) \to HK^{p+1}(A, M)$$

$$\overline{e}_A \underset{K}{\frown} - : HK_q(A, M) \to HK_{q-1}(A, M)$$

are differentials, defining higher (co)homology spaces

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

$$\overline{e}_A \underset{K}{\smile} - : HK^p(A, M) \to HK^{p+1}(A, M)$$

$$\overline{e}_A \underset{K}{\frown} - : HK_q(A, M) \to HK_{q-1}(A, M)$$

are differentials, defining higher (co)homology spaces

$$HK^p_{hi}(A, M) \text{ and } HK^{hi}_q(A, M).$$

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

$$\overline{e}_A \underset{K}{\smile} - : HK^p(A, M) \to HK^{p+1}(A, M)$$

$$\overline{e}_A \underset{K}{\frown} - : HK_q(A, M) \to HK_{q-1}(A, M)$$

are differentials, defining higher (co)homology spaces

$$HK^p_{hi}(A, M) \text{ and } HK^{hi}_q(A, M).$$

Then the higher Koszul calculus consists of the graded associative algebra $(HK^\bullet_{hi}(A), \underset{K}{\smile})$

Higher Koszul (co)homology

From $e_A \underset{K}{\smile} e_A = 0$, we draw $\overline{e}_A \underset{K}{\smile} \overline{e}_A = 0$. Therefore

$$\overline{e}_A \underset{K}{\smile} - : HK^p(A, M) \to HK^{p+1}(A, M)$$

$$\overline{e}_A \underset{K}{\frown} - : HK_q(A, M) \to HK_{q-1}(A, M)$$

are differentials, defining higher (co)homology spaces

$$HK^p_{hi}(A, M) \text{ and } HK^{hi}_q(A, M).$$

Then the higher Koszul calculus consists of the graded associative algebra $(HK^\bullet_{hi}(A), \underset{K}{\smile})$

and of the graded $HK^\bullet_{hi}(A)$-bimodules $(HK^\bullet_{hi}(A, M)$ and $(HK^{hi}_\bullet(A, M)$ for actions $\underset{K}{\smile}$ and $\underset{K}{\frown}$ respectively.

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation. The Rinehart-Goodwillie identity

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation.
The Rinehart-Goodwillie identity

$$[H(D \frown -), H(B)]_c = H(L_D)$$

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation. The Rinehart-Goodwillie identity

$$[H(D \frown -), H(B)]_c = H(L_D)$$

on $HH_\bullet(A)$ shows that, whenever $char(k) = 0$, $HH_0^{hi}(A) \cong k$ and $HH_p^{hi}(A) \cong 0$ if $p > 0$.

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation. The Rinehart-Goodwillie identity

$$[H(D \frown -), H(B)]_c = H(L_D)$$

on $HH_\bullet(A)$ shows that, whenever $char(k) = 0$, $HH_0^{hi}(A) \cong k$ and $HH_p^{hi}(A) \cong 0$ if $p > 0$.
Consequently, we have the following theorem.

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation. The Rinehart-Goodwillie identity

$$[H(D \frown -), H(B)]_c = H(L_D)$$

on $HH_\bullet(A)$ shows that, whenever $char(k) = 0$, $HH_0^{hi}(A) \cong k$ and $HH_p^{hi}(A) \cong 0$ if $p > 0$.
Consequently, we have the following theorem.

Theorem (BLS). Assume $char(k) = 0$. If A is quadratic and Koszul, then

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation. The Rinehart-Goodwillie identity

$$[H(D \frown -), H(B)]_c = H(L_D)$$

on $HH_\bullet(A)$ shows that, whenever $char(k) = 0$, $HH_0^{hi}(A) \cong k$ and $HH_p^{hi}(A) \cong 0$ if $p > 0$.
Consequently, we have the following theorem.

Theorem (BLS). Assume $char(k) = 0$. If A is quadratic and Koszul, then $HK_0^{hi}(A) \cong k$ and $HK_p^{hi}(A) \cong 0$ if $p > 0$.

A noncommutative Poincaré's Lemma for graded algebras

Here A is any connected $\mathbb{N}$-graded k-algebra, D is its Euler derivation. The Rinehart-Goodwillie identity

$$[H(D \frown -), H(B)]_c = H(L_D)$$

on $HH_\bullet(A)$ shows that, whenever $char(k) = 0$, $HH_0^{hi}(A) \cong k$ and $HH_p^{hi}(A) \cong 0$ if $p > 0$.
Consequently, we have the following theorem.

Theorem (BLS). Assume $char(k) = 0$. If A is quadratic and Koszul, then $HK_0^{hi}(A) \cong k$ and $HK_p^{hi}(A) \cong 0$ if $p > 0$.

Question : is this theorem still valid when $N > 2$?

Why is this quadratic theorem important ?

The quadratic algebra $A = k\langle x, y\rangle / \langle x^2, y^2 - xy\rangle$ is not Koszul and we have proved that $HK_2^{hi}(A) \neq 0$ (actually 2-dimensional).

Why is this quadratic theorem important ?

The quadratic algebra $A = k\langle x, y\rangle/\langle x^2, y^2 - xy\rangle$ is not Koszul and we have proved that $HK_2^{hi}(A) \neq 0$ (actually 2-dimensional).

Then for this quadratic algebra A, the higher Koszul homology provides more information on A than the higher Hochschild homology.

Why is this quadratic theorem important ?

The quadratic algebra $A = k\langle x, y\rangle / \langle x^2, y^2 - xy\rangle$ is not Koszul and we have proved that $HK_2^{hi}(A) \neq 0$ (actually 2-dimensional).

Then for this quadratic algebra A, the higher Koszul homology provides more information on A than the higher Hochschild homology.

We conjecture that this fact holds for any quadratic algebra, and why not, for any N-homogeneous algebra.

Why is this quadratic theorem important ?

The quadratic algebra $A = k\langle x, y\rangle/\langle x^2, y^2 - xy\rangle$ is not Koszul and we have proved that $HK_2^{hi}(A) \neq 0$ (actually 2-dimensional).

Then for this quadratic algebra A, the higher Koszul homology provides more information on A than the higher Hochschild homology.

We conjecture that this fact holds for any quadratic algebra, and why not, for any N-homogeneous algebra.

The same questions can be asked about the following theorem.

Why is this quadratic theorem important ?

The quadratic algebra $A = k\langle x, y\rangle/\langle x^2, y^2 - xy\rangle$ is not Koszul and we have proved that $HK_2^{hi}(A) \neq 0$ (actually 2-dimensional).

Then for this quadratic algebra A, the higher Koszul homology provides more information on A than the higher Hochschild homology.

We conjecture that this fact holds for any quadratic algebra, and why not, for any N-homogeneous algebra.

The same questions can be asked about the following theorem.

Theorem (BLS). Assume $char(k) = 0$. If A is quadratic, Koszul and n-Calabi-Yau, then $HK^n_{hi}(A) \cong k$ and $HK^p_{hi}(A) \cong 0$ if $p \neq n$.