Weakly Confluent Quadratic Algebras Computations of Section 4

Roland Berger

LARAL, Faculté des Sciences et Techniques
23, Rue P. Michelon, 42023 Saint-Etienne Cedex, France
E-mail address: Roland.Berger@univ-st-etienne.fr

Abstract

The subject of Section 4 of my paper: Weakly Confluent Quadratic Algebras, *Algebras and Representation Theory* **1** (1998), 189-213, is a classification of quadratic algebras with two generators. The case-by-case computations of this classification are performed here in details. We keep the notations of the paper.

In the sequel, we have $r = 2$ and the ordered basis X of V is $x < y$. The quadratic algebra A over V has E as space of relations and is described by the X-reduction operator S such that $Ker(S) = E$. Our purpose is firstly to decide the X-confluence of A, i.e. the confluence of S, and next to compute the left $B_X(V)$-classes making up $C(S)$. The different cases we have to examine are specified by $dim(S)$ and by the set of S-reduced monomials. Once this set is fixed, S is evaluated on the nonreduced monomials with the aid of parameters. Obviously the confluence of S and the determination of $C(S)$ depend on the parameters. It is worth remarking that the computation of $C(S)$ uses in some places the following hypothesis on the ground field K:

(H) Any polynomial of degree two in one indeterminate over K has a root in K.

When $C(S) = \emptyset$, we shall examine the Koszulity of A. Actually, under (H), we shall see that A is never Koszul in this case. This means that weak confluence and Koszulity are equivalent under $r = 2$ and (H). The following notations will be useful. We denote by P the pair (S_1, S_2). If M is a set of monomials, $< M >$ is the vector subspace spanned by M. On the other hand, ϕ refers to a generic element of $GL(V)$ defined by $\phi(x) = ax + by$ and $\phi(y) = cx + dy$ with a, b, c, d in K. For brevity, we let $T = \phi.S$. Notice that if S is confluent, $C(S)$ contains $B_X(V)$, i.e. the ϕ such that $b = 0$.

Case $\mathbf{dim(S)} = \mathbf{3}$. *Subcase 1.* $Nred(S) = <xx>$. Then S is trivial, hence confluent. The subspace $Ker(T)$ is spanned by $b^2yy + abyx + abxy + a^2xx$. If $b = 0$, $T = S$ is confluent. Otherwise, $Nred(T) = <yy>$ and

$$T(yy) = -\frac{a^2}{b^2}xx - \frac{a}{b}(xy + yx)\,. \tag{0.1}$$

The subcase 4 below shows that T is confluent. Thus $C(S) = GL(V)$.

Subcase 2. $Nred(S) = <xy>$. Then $S(xy) = \alpha xx$. There is no P-ambiguous monomial, hence S is confluent. The subspace $Ker(T)$ is spanned by the following polynomial

$$b(d - \alpha b)yy + b(c - \alpha a)yx + a(d - \alpha b)xy + a(c - \alpha a)xx\,.$$

We can assume that $b \neq 0$. If $d \neq \alpha b$, we have $Nred(T) = <yy>$ and

$$T(yy) = \alpha' xx + \beta' xy + \gamma' yx\,,\ \alpha' = \frac{a(\alpha a - c)}{b(d - \alpha b)}\,,\ \beta' = -\frac{a}{b}\,,\ \gamma' = \frac{\alpha a - c}{d - \alpha b}\,. \tag{0.2}$$

As $\beta' \neq \gamma'$, the subcase 4 below shows that T is not confluent. If $d = \alpha b$, then $Nred(T) = <yx>$ and the subcase 3 below shows that T is confluent. Thus $C(S)$ is the union of two left $B_X(V)$-classes: $B_X(V)$ and $B_X(V)\psi$ where the matrix of ψ relatively to X is the following

$$\begin{pmatrix} 1 & 0 \\ 1 & \alpha \end{pmatrix}\,. \tag{0.3}$$

Subcase 3. $Nred(S) = <yx>$. Then $S(yx) = \alpha xx + \beta xy$. There is no P-ambiguous monomial, hence S is confluent and we can assume $b \neq 0$. The subspace $Ker(T)$ is spanned by the following polynomial

$$b((1-\beta)d-\alpha b)yy+(ad-\alpha ab-\beta bc)yx+(bc-\alpha ab-\beta ad)xy+(ac-\alpha a^2-\beta ac)xx\,.$$

First assume $(1 - \beta)d \neq \alpha b$. Then $Nred(T) = <yy>$ and we can write down

$$T(yy) = \alpha' xx + \beta' xy + \gamma' yx\,. \tag{0.4}$$

The condition $\beta' = \gamma'$ implies $\beta = -1$, but an easy calculation shows that the condition $\alpha' + \beta'^2 = 0$ does not hold in this case. Therefore T is not confluent.

Assuming next $(1-\beta)d = \alpha b$ and rewriting the above polynomial, we see that T falls in the subcases 2 or 3, so T is confluent. Thus the matrices of the elements of $C(S)$ not in $B_X(V)$ are the following

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ with } b \neq 0\,,\ (1-\beta)d = \alpha b\,,\ ad - bc \neq 0\,. \qquad (0.5)$$

The results obtained so far can be summarized in the following way:

(i) If $\beta = 1$ and $\alpha = 0$ (i.e. $A = Sym(V)$), $C(S) = GL(V)$.

(ii) If $\beta = 1$ and $\alpha \neq 0$, $C(S) = B_X(V)$.

(iii) If $\beta \neq 1$, $C(S)$ is the union of $B_X(V)$ and $B_X(V)\psi$, where the matrix of ψ relatively to X is the following

$$\begin{pmatrix} 1 & 0 \\ 1 & \frac{\alpha}{1-\beta} \end{pmatrix}. \qquad (0.6)$$

Subcase 4. $Nred(S) =< yy >$. Then $S(yy) = \alpha xx + \beta xy + \gamma yx$. The single P-ambiguous monomial is yyy. Comparing

$$R_1^P(yyy) = \alpha\beta xxx + (\alpha + \beta^2)xxy + \beta\gamma xyx + \gamma yxy$$

with

$$R_2^P(yyy) = \alpha\gamma xxx + \beta\gamma xyx + (\alpha + \gamma^2)yxx + \beta yxy$$

we get the following equivalence

$$S \text{ confluent} \Longleftrightarrow \beta = \gamma \text{ and } \alpha + \beta^2 = 0\,. \qquad (0.7)$$

(i) Assume that S is confluent, that is $S(yy) = -\beta^2 xx + \beta(xy + yx)$. The subspace $Ker(T)$ is spanned by the following polynomial

$$(d-\beta b)^2 yy + (c-\beta a)(d-\beta b)(yx + xy) + (c-\beta a)^2 xx\,.$$

If $d = \beta b$, T is confluent. Otherwise, $Nred(T) =< yy >$ and

$$T(yy) = -\frac{(c-\beta a)^2}{(d-\beta b)^2} xx + \frac{\beta a - c}{d - \beta b}(xy + yx)\,. \qquad (0.8)$$

Therefore T is confluent. Thus $C(S) = GL(V)$.

(ii) Assume that S is not confluent. The case $Nred(T) =< yy >$ occurs if and only if $d^2 - (\beta + \gamma)bd - \alpha b^2$ does not vanish. But in this case, (i) shows that T is not confluent. On the other hand, if $Nred(T) \neq < yy >$, T is confluent and the equality $C(S) = C(T)\phi$ allows us to deduce $C(S)$ from the computations of previous subcases. Our strategy is now to find T such that

$Nred(T) \neq < yy >$. Note that non-confluence of S means that β is not the unique root of the following equation (with unknown d):

$$d^2 - (\beta + \gamma)d - \alpha = 0 \,. \tag{0.9}$$

Under the hypothesis (H), we can choose a root $d \neq \beta$ of (0.9) (without (H), it suffices to assume that (0.9) has a root in K). Taking $a = 0$, $b = c = 1$ in ϕ, we get $Nred(T) = < yx >$ and

$$T(yx) = \alpha' xx + \beta' xy \,, \; \alpha' = \frac{1}{\beta - d} \,, \; \beta' = \frac{d - \gamma}{\beta - d} \,. \tag{0.10}$$

Thus if $\beta' = 1$, $C(S) = B_X(V)\phi$ and if $\beta' \neq 1$, $C(S)$ is the union of $B_X(V)\phi$ and $B_X(V)\psi$, where the matrix of ψ relatively to X is the following

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{\beta+\gamma-2d} & \frac{\beta+\gamma-d}{\beta+\gamma-2d} \end{pmatrix} . \tag{0.11}$$

If (0.9) has no root in K, then $C(S) = \emptyset$ (use an extension of K in which (0.9) has a root).

Case $\mathbf{dim(S) = 1}$. We deduce this case from the previous one by duality. Applying the kernel, the following is easily checked

$$(\phi . S)^! = (\phi^{-1*}) . S^! \,. \tag{0.12}$$

Therefore $\phi \mapsto \phi^{-1*}$ is one-to-one between $C(S)$ and $C(S^!)$. Moreover if S runs over the present case, $S^!$ runs over the previous case. We just give the results and leave the verifications to the reader.

Subcase 1. $Red(S) = <yy>$. S is confluent and $C(S) = GL(V)$.

Subcase 2. $Red(S) = < yx >$. Then $S(yy) = \alpha yx$, S is confluent and $C(S)$ is the same as in subcase 2 of the previous case.

Subcase 3. $Red(S) = < xy >$. Then $S(yy) = \alpha xy$ and $S(yx) = \beta xy$, S is confluent and we have

(i) If $\beta = -1$ and $\alpha = 0$ (i.e. $A = \Lambda(V)$), $C(S) = GL(V)$.

(ii) If $\beta = -1$ and $\alpha \neq 0$, $C(S) = B_X(V)$.

(iii) If $\beta \neq -1$, $C(S)$ is the union of $B_X(V)$ and $B_X(V)\psi$, where the matrix of ψ relatively to X is the following

$$\begin{pmatrix} 1 & 0 \\ 1 & \frac{\alpha}{1+\beta} \end{pmatrix} . \tag{0.13}$$

Subcase 4. $Red(S) = <xx>$. Then $S(yy) = \alpha xx$, $S(yx) = \beta xx$ and $S(xy) = \gamma xx$. Furthermore S is confluent if and only if

$$\beta = \gamma \text{ and } \alpha = \beta^2 , \tag{0.14}$$

and $C(S) = GL(V)$ in this case. If S is not confluent, take $a = 0$, $b = c = 1$ and $d \neq \beta$ root of $d^2 - (\beta + \gamma)d + \alpha = 0$ (d exists under (H)). Then T falls in subcase 3 and $C(S) = C(T)\phi$. Moreover $T(yy) = \alpha' xy$ and $T(yx) = \beta' xy$, with $\alpha' = \frac{1}{\beta - d}$ and $\beta' = \frac{\gamma - d}{\beta - d}$. If $\beta' = -1$, $C(S) = B_X(V)\phi$. If $\beta' \neq -1$, $C(S)$ is the union of $B_X(V)\phi$ and $B_X(V)\psi$, where the matrix of ψ is the same as in (0.11).

Case dim(S) = 2. *Subcase 1.* $Red(S) = <yx, yy>$. S is trivial hence confluent. The subspace $Ker(T)$ is spanned by $u = bdyy + adyx + bcxy + acxx$ and $v = d^2yy + cdyx + cdxy + c^2xx$. If $d \neq 0$, then $Red(T) = <xx, xy>$ and

$$T(yy) = -\frac{c}{d}xy\,,\ T(yx) = -\frac{c}{d}xx\,. \tag{0.15}$$

Subcase 6 below shows that T is confluent. If $d = 0$, $Red(T) = <xy, yy>$ and T is trivial, hence confluent. Thus $C(S) = GL(V)$.

Subcase 2. $Red(S) = <xy, yy>$. Then $S(yx) = \alpha xy$, $S(xx) = 0$. The single P-ambiguous monomial is yxx. Since $R_1^P(yxx) = R_2^P(yxx) = 0$, S is confluent and we can assume $b \neq 0$. The subspace $Ker(T)$ is spanned by the following polynomials

$$u = (1-\alpha)bdyy + (ad - \alpha bc)yx + (bc - \alpha ad)xy + (1-\alpha)acxx\,,$$

$$v = b^2yy + abyx + abxy + a^2xx\,.$$

Putting the relation $yy = -\frac{a}{b}yx - \frac{a}{b}xy - \frac{a^2}{b^2}xx$ in u, we get the following $\alpha yx - xy - (1-\alpha)\frac{a}{b}xx = 0$. Assuming $\alpha \neq 0$, we have $Red(T) = <xx, xy>$ and

$$\begin{cases} T(yy) = -\frac{a^2}{\alpha b^2}xx - \frac{(1+\alpha)a}{\alpha b}xy \\ T(yx) = \frac{(1-\alpha)a}{\alpha b}xx + \frac{1}{\alpha}xy\,. \end{cases} \tag{0.16}$$

Using subcase 6 below and some calculations, one checks that T is confluent. Next assume $\alpha = 0$. Then $Red(T) = <xx, yx>$ and

$$\begin{cases} T(yy) = -\frac{a}{b}yx \\ T(xy) = -\frac{a}{b}xx\,. \end{cases} \tag{0.17}$$

Subcase 5 shows that T is confluent. Thus $C(S) = GL(V)$.

Subcase 3. $Red(S) = <xx, yy>$. Then $S(yx) = \alpha xx$, $S(xy) = \beta xx$. The P-ambiguous monomials are xyx and yxy. From $R_1^P(xyx) = \beta xxx$, $R_2^P(xyx) = \alpha xxx$, $R_1^P(yxy) = R_2^P(yxy) = \alpha\beta xxx$, we obtain

$$S \text{ confluent} \Longleftrightarrow \alpha = \beta\,. \tag{0.18}$$

(i) Assume S confluent, i.e. $S(yx) = S(xy) = \alpha xx$. $Ker(T)$ is spanned by

$$u = b(d-\alpha b)yy + a(d-\alpha b)yx + b(c-\alpha a)xy + a(c-\alpha a)xx\,,$$

$$v = b(d-\alpha b)yy + b(c-\alpha a)yx + a(d-\alpha b)xy + a(c-\alpha a)xx\,.$$

We can assume $b \neq 0$. If $d \neq \alpha b$, one checks that $Red(T) = <xx, xy>$ and

$$\begin{cases} T(yy) = \frac{a(\alpha a-c)}{b(d-\alpha b)}xx + \frac{2\alpha ab-ad-bc}{d-\alpha b}xy \\ T(yx) = xy\,. \end{cases} \tag{0.19}$$

Subcase 6 below shows that T is confluent. If $d = \alpha b$, we have $u = b(c-\alpha a)xy + a(c-\alpha a)xx$ and $v = b(c-\alpha a)yx + a(c-\alpha a)xx$. Then $Red(T) = <xx, yy>$ and

$$T(yx) = T(xy) = -\frac{a}{b}xx\,. \tag{0.20}$$

So T is confluent. We conclude that $C(S) = GL(V)$.

(ii) Assume that S is not confluent. *Claim*: A is not 4-Koszul, i.e. the sublattice of $\mathcal{L}_X(V^{(4)})$ generated by the X-reduction operators S_1, S_2, S_3 is *not* distributive. Thanks to Priddy's theorem, this claim implies that $C(S) = \emptyset$. Before proving the claim, we need a lemma which relates 4-Koszulity to confluence defect. In this lemma (and the following), r is arbitrary.

Lemma 0.1 *The algebra A is 4-Koszul if and only if the following holds*

$$2r\, def_c(P) = def_c(S_2, S_1 \wedge S_3) + def_c(S_2, S_1 \vee S_3)\,. \tag{0.21}$$

On the right-hand side of (0.21), the reduction operators are considered as defined on $V^{(4)}$. Clearly, Lemma 0.1 is just an application of the following general lemma.

Lemma 0.2 *The sublattice of $\mathcal{L}_X(V)$ generated by the X-reduction operators S, T, U is distributive if and only if we have*

$$def_c(S,T) + def_c(S,U) + def_c(T,U) = dim(Obs_{red}^{S,T,U}) + dim(Obs_{amb}^{S,T,U})\,. \tag{0.22}$$

Moreover, if S and U are confluent, (0.22) is replaced by the following

$$def_c(S,T) + def_c(T,U) = def_c(T, S \wedge U) + def_c(T, S \vee U)\,. \tag{0.23}$$

Proof. Recall the following relations [1]

$$\begin{cases} Red(S)\cap Red(T)\cap Red(U)=Red(S\wedge T\wedge U)\oplus Obs_{red}^{S,T,U} \\ Red(S\vee T\vee U)=(Red(S)+Red(T)+Red(U))\oplus Obs_{amb}^{S,T,U}\,. \end{cases} \tag{0.24}$$

Take the dimension in the second relation (0.24). To compute the dimension of $Red(S)+Red(T)+Red(U)$, use the principle of inclusion-exclusion, and next take the dimension in the first relation (0.24). Then the expression

$$dim S\vee T\vee U - dim S - dim T - dim U + dim S\wedge T + dim S\wedge U + dim T\wedge U - dim S\wedge T\wedge U$$

is equal to the following

$$dim(Obs_{red}^{S,T,U}) + dim(Obs_{amb}^{S,T,U}) - def_c(S,T) - def_c(S,U) - def_c(T,U).$$

But it is well-known that S, T, U generate a distributive sublattice if and only if the former expression vanishes [2]. Thus the first assertion is clear. On the other hand, recall the following relations [1]

$$\begin{cases} Obs_{red}^{S,T,U} = Obs_{red}^{T,S\wedge U}\oplus (Nred(T)\cap Obs_{red}^{S,U}) \\ Obs_{amb}^{S,T,U} = Obs_{amb}^{T,S\vee U}\oplus (Nred(T)\cap Obs_{amb}^{S,U})\,. \end{cases} \tag{0.25}$$

Using that, (0.22) becomes (0.23) when S and U are confluent. ■

We return to the proof of the claim. Clearly, xyx is the single ambiguous P-obstruction. In particular, $def_c(P)=1$. For computing the right-hand side of (0.21), note that $S_1\wedge S_3=S\otimes S$ and $S_1\vee S_3=S_1+S_3-S\otimes S$. The $(S_2, S_1\wedge S_3)$-ambiguous monomials are the following: $xxyx$, $xyxx$, $xyxy$, $yxyx$, $yxyy$, $yyxy$. One easily checks that $xxyx$ is the single obstruction, hence $def_c(S_2, S_1\wedge S_3)=1$. On the other hand, the $(S_2, S_1\vee S_3)$-ambiguous monomials are $xyxy$, $yxyx$. It is immediate that they are both obstructions, hence $def_c(S_2, S_1\vee S_3)=2$. Thus (0.21) does not hold, as we claimed.

Subcase 4. $Red(S)=<xy, yx>$. Then $S(yy)=\alpha xy+\beta yx$, $S(xx)=0$. This subcase is dual to the previous one, and we have $S^!(x^*y^*)=-\alpha y^*y^*$, $S^!(y^*x^*)=-\beta y^*y^*$. Therefore the equivalence (0.18) holds again. Furthermore if S is confluent, $C(S)=GL(V)$. Otherwise, S is not 4-Koszul and $C(S)=\emptyset$.

Subcase 5. $Red(S)=<xx, yx>$. Then $S(yy)=\alpha xx+\beta yx$, $S(xy)=\gamma xx$. The P-ambiguous monomials are xyy and yyy. We have

$$\begin{cases} R_1^P(xyy)=\gamma^2 xxx \\ R_2^P(xyy)=(\alpha+\beta\gamma)xxx \end{cases} \quad \begin{cases} R_1^P(yyy)=\alpha\gamma xxx+\beta\gamma yxx \\ R_2^P(yyy)=\alpha\beta xxx+(\alpha+\beta^2)yxx \end{cases} \tag{0.26}$$

Thus S is confluent if and only if

$$\alpha = 0 \text{ and } \beta = \gamma\,. \tag{0.27}$$

(i) S is confluent, i.e. $S(yy) = \beta yx$, $S(xy) = \beta xx$. We can assume $b \neq 0$. $Ker(T)$ is spanned by the following

$$u = d(d-\beta b)yy + d(c-\beta a)yx + c(d-\beta b)xy + c(c-\beta a)xx\,,$$

$$v = b(d-\beta b)yy + b(c-\beta a)yx + a(d-\beta b)xy + a(c-\beta a)xx\,.$$

If $d \neq \beta b$, we have $Red(T) = <xx, yx>$ and

$$\begin{cases} T(yy) = \frac{\beta a - c}{d - \beta b} yx \\ T(xy) = \frac{\beta a - c}{d-\beta b} xx\,. \end{cases} \tag{0.28}$$

(0.27) shows that T is confluent. If $d = \beta b$, then T is trivial such that $Red(T) = <xy, yy>$. We conclude that $C(S) = GL(V)$.

(ii) S is not confluent. Our aim is to prove that S is not 4-Koszul, which implies $C(S) = \emptyset$. We shall use Lemma 0.1. From (0.26), it is easily checked that $def_c(P) = 2$ if $\alpha + \beta\gamma \neq \gamma^2$ and $\alpha + \beta^2 \neq \beta\gamma$, and $def_c(P) = 1$ otherwise (S non-confluent means that the equalities $\alpha + \beta\gamma = \gamma^2$ and $\alpha + \beta^2 = \beta\gamma$ do not simultaneously occur).

Assume $\alpha + \beta\gamma \neq \gamma^2$. The reader can verify that the ambiguous $(S_2, S_1 \wedge S_3)$-obstructions are $xxyy$ and $yxyy$. In particular, $def_c(S_2, S_1 \wedge S_3) = 2$. Furthermore the $(S_2, S_1 \vee S_3)$-ambiguous monomials are $xyyy$, $yyyy$. The latter are both obstructions if $\alpha + \beta^2 \neq \beta\gamma$, and the single obstruction is $xyyy$ if $\alpha + \beta^2 = \beta\gamma$. Thus (0.21) does not hold in any case.

If $\alpha + \beta\gamma = \gamma^2$, we argue by duality. As a matter of fact, $S^!$ is also in Subcase 5, but $S^!(x^*x^*) = \alpha' y^*y^* + \beta' x^*y^*$, $S^!(y^*x^*) = \gamma' y^*y^*$ with $\alpha' = -\alpha$, $\beta' = -\gamma$, $\gamma' = -\beta$. So we recover the previous case $\alpha' + \beta'\gamma' \neq \gamma'^2$ and the same conclusion.

Subcase 6. $Red(S) = <xx, xy>$. Then $S(yy) = \alpha xx + \beta xy$, $S(yx) = \gamma xx + \delta xy$. The P-ambiguous monomials are yyx and yyy. From

$$\begin{cases} R_1^P(yyx) = (\beta\gamma + \alpha)xxx + \beta\delta xxy \\ R_2^P(yyx) = (\alpha\delta^2 + \gamma^2\delta + \gamma^2)xxx + (\beta\delta^2 + \gamma\delta^2 + \gamma\delta)xxy\,, \end{cases} \tag{0.29}$$

$$\begin{cases} R_1^P(yyy) = \alpha\beta xxx + (\beta^2 + \alpha)xxy \\ R_2^P(yyy) = (\alpha\beta\delta + \alpha\gamma\delta + \alpha\gamma)xxx + (\alpha\delta^2 + \beta^2\delta + \beta\gamma)xxy\,, \end{cases} \tag{0.30}$$

it is easily proved that confluence of S is equivalent to the following system

$$\left\{ \begin{array}{l} \beta(1-\delta) = \gamma(1+\delta) \\ \alpha(1-\delta^2) = -\beta\gamma\delta\,. \end{array} \right. \tag{0.31}$$

The set of solutions is divided in three parts: (1) $\delta = 1$ and $\gamma = 0$, (2) $\delta = -1$ and $\beta = 0$, (3) $\delta \neq \pm 1$ and $\alpha = -\frac{\gamma^2\delta}{(1-\delta)^2}$, $\beta = \frac{\gamma(1+\delta)}{1-\delta}$.

The subspace $Ker(T)$ is spanned by the following

$$u = (d^2 - \alpha b^2 - \beta bd)yy + (cd - \alpha ab - \beta bc)yx + \cdots\,,$$

$$v = (bd - \gamma b^2 - \delta bd)yy + (ad - \gamma ab - \delta bc)yx + \cdots\,.$$

Denote by D the determinant of u and v (truncated as above) w.r.t. the basis (yy, yx) and check that $D = det(\phi)[d^2 - (\beta+\gamma)bd - (\alpha\delta - \beta\gamma)b^2]$. *Claim*: $C(S) = GL(V)$ if S is confluent, $C(S) = \emptyset$ and S is not 4-Koszul if S is not confluent. We begin by proving the claim when K satisfies the hypothesis (H). In fact, we can choose $a = 0$, $b = c = 1$ and d as a root of the following equation:

$$d^2 - (\beta+\gamma)d - (\alpha\delta - \beta\gamma) = 0\,. \tag{0.32}$$

Then T does not fall in Subcase 6. Using $C(S) = C(T)\phi$ and the fact that the claim holds in the previous subcases, we see that the claim holds in our present subcase. When K does not satisfy (H), the claim is still valid by considering a suitable algebraic extension of K.

It is convenient to sum up some results obtained during the preceding discussion in two theorems. It is worth remarking that (H) is not used in the second theorem, nor in *(b)* of the first one (we do not know if (H) is essential for *(c)*).

Theorem 0.3 *Suppose the ground field K verifies (H). Suppose A is a quadratic algebra in two generators and denote by E its space of relations.*

(a) If $dimE$ is one or three, A is weakly confluent.

(b) If $dimE$ is two, the following are equivalent: (i) A is weakly confluent, (ii) A is strongly confluent, (iii) A is 4-Koszul, (iv) A is Koszul.

(c) A is weakly confluent if and only if A is Koszul (or 4-Koszul).

Theorem 0.4 *The strongly confluent quadratic algebras A in two generators x and y are the following (we list the bases of E; greek letters are parameters in K):*

$dimE = 0$: $\emptyset$.

$dimE = 1$: xx; $yx - xy$; $yy + \lambda^2 xx - \lambda xy - \lambda yx$.

$dimE = 2$: (xy, xx); $(yx - \lambda xy, xx)$; $(yx - \lambda xx, xy - \lambda xx)$; $(yy - \lambda xy - \lambda yx, xx)$; $(yy - \lambda yx, xy - \lambda xx)$; $(yy - \alpha xx - \beta xy, yx - \gamma xx - \delta xy)$ with (0.31).

$dimE = 3$: (yx, xy, xx); $(yy, yx + xy, xx)$; $(yy - \lambda^2 xx, yx - \lambda xx, xy - \lambda xx)$.

$dimE = 4$: (yy, yx, xy, xx).

References

[1] R. Berger, Confluence and Koszulity, *J. Algebra* **201** (1998), 243-283.

[2] G. Birkhoff, Lattice theory (3rd edn), A.M.S., Providence RI, 1967.