HOMOGENEOUS ALGEBRAS

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Abstract

Various concepts associated with quadratic algebras admit natural generalizations when the quadratic algebras are replaced by graded algebras which are finitely generated in degree 1 with homogeneous relations of degree N. Such algebras are referred to as *homogeneous algebras of degree* N. In particular it is shown that the Koszul complexes of quadratic algebras generalize as N-complexes for homogeneous algebras of degree N.

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1 Introduction and Preliminaries

Our aim is to generalize the various concepts associated with quadratic algebras as described in [27] when the quadratic algebras are replaced by the homogeneous algebras of degree N with $N \geq 2$ ($N = 2$ is the case of quadratic algebras). Since the generalization is natural and relatively straightforward, the treatment of [26], [27] and [25] will be directly adapted to homogeneous algebras of degree N. In other words we dispense ourselves to give a review of the case of quadratic algebras (i.e. the case $N = 2$) by referring to the above quoted nice treatments. In proceeding to this adaptation, we shall make use of the following slight elaboration of an ingredient of the elegant presentation of [25].

LEMMA 1 *Let A be an associative algebra with product denoted by m, let C be a coassociative coalgebra with coproduct denoted by Δ and let $Hom_{\mathbb{K}}(C, A)$ be equipped with its structure of associative algebra for the convolution product $(\alpha, \beta) \mapsto \alpha * \beta = m \circ (\alpha \otimes \beta) \circ \Delta$. Then one defines an algebra-homomorphism $\alpha \mapsto d_\alpha$ of $Hom_{\mathbb{K}}(C, A)$ into the algebra $End_A(A \otimes C) = Hom_A(A \otimes C, A \otimes C)$ of endomorphisms of the left A-module $A \otimes C$ by defining d_α as the composite*

$$A \otimes C \stackrel{I_A \otimes \Delta}{\longrightarrow} A \otimes C \otimes C \stackrel{I_A \otimes \alpha \otimes I_C}{\longrightarrow} A \otimes A \otimes C \stackrel{m \otimes I_C}{\longrightarrow} A \otimes C$$

for $\alpha \in Hom_{\mathbb{K}}(C, A)$.

The proof is straightforward, $d_\alpha \circ d_\beta = d_{\alpha * \beta}$ follows easily from the coassociativity of Δ and the associativity of m. As pointed out in [25] one obtains a graphical version ("electronic version") of the proof by using the usual graphical version of the coassociativity of Δ combined with the usual graphical version of the associativity of m. The left A-linearity of d_α is straightforward.

In the above statement as well as in the following, all vector spaces, algebras, coalgebras are over a fixed field $\mathbb{K}$. Furthermore unless otherwise specified the algebras are unital associative and the coalgebras are counital coassociative. For instance in the previous case, if $\mathbb{1}$ is the unit of A and ε is the counit of C, then the unit of $\mathrm{Hom}_{\mathbb{K}}(C, A)$ is the linear mapping $\alpha \mapsto \varepsilon(\alpha)\mathbb{1}$ of C into A. In Lemma 1 the left A-module structure on $A \otimes C$ is the obvious one given by

$$x(a \otimes c) = (xa) \otimes c$$

for any $x \in A$, $a \in A$ and $c \in C$.

Besides the fact that it is natural to generalize for other degrees what exists for quadratic algebras, this paper produces a very natural class of N-complexes which generalize the Koszul complexes of quadratic algebras [26], [27], [33], [25], [19] and which are not of simplicial type. By N-complexes of simplicial type we here mean N-complexes associated with simplicial modules and N-th roots of unity in a very general sense [12] which cover cases considered e.g. in [28], [20], [16], [11], [21] the generalized homology of which has been shown to be equivalent to the ordinary homology of the corresponding simplicial modules [12]. This latter type of constructions and results has been recently generalized to the case of cyclic modules [35]. In spite of the fact that they compute the ordinary homology of the simplicial modules, the usefulness of these N-complexes of simplicial type comes from the fact that they can be combined with other N-complexes [17], [18]. In fact the BRS-like construction [4] of [18] shows that spectral sequences arguments (e.g. in the form of a generalization of the homological perturbation theory [31]) are still working for N-complexes. Other nontrivial classes of N-complexes which are not of simplicial type are the universal construction of [16] and the

N-complexes of [14], [15] (see also in [13] for a review). It is worth noticing here that elements of homological algebra for N-complexes have been developed in [21] and that several results for N-complexes and more generally N-differential modules like Lemma 1 of [12] have no nontrivial counterpart for ordinary complexes and differential modules. It is also worth noticing that besides the above mentioned examples, various problems connected with theoretical physics implicitly involve exotic N-complexes (see e.g. [23], [24]).

In the course of the paper we shall point out the possibility of generalizing the approach based on quadratic algebras of [27] to quantum spaces and quantum groups by replacing the quadratic algebras by N-homogeneous ones. Indeed one also has in this framework internal **end**, etc. with similar properties.

Finally we shall revisit in the present context the approach of [8], [9] to Koszulity for N-homogeneous algebras. This is in order since as explained below, the generalization of the Koszul complexes introduced in this paper for N-homogeneous algebras is a canonical one. We shall explain why a definition based on the acyclicity of the N-complex generalizing the Koszul complex is inappropriate and we shall identify the ordinary complex introduced in [8] (the acyclicity of which is the definition of Koszulity of [8]) with a complex obtained by contraction from the above Koszul N-complex. Furthermore we shall show the uniqueness of this contracted complex among all other ones. Namely we shall show that the acyclicity of any other complex (distinct from the one of [8]) obtained by contraction of the Koszul N-complex leads for $N \geq 3$ to an uninteresting (trivial) class of algebras.

Some examples of Koszul homogeneous algebras of degree > 2 are given in [8], including a certain cubic Artin-Schelter regular algebra [1]. Recall that Koszul quadratic algebras arise in several topics as algebraic geometry [22], representation theory [5], quantum groups [26], [27], [33], [34], Sklyanin algebras [30], [32]. A classification of the Koszul quadratic algebras with two generators over the complex numbers is performed in [7]. Koszulity of non-quadratic algebras and each of the above items deserve further attention.

The plan of the paper is the following.
In Section 2 we define the duality and the two (tensor) products which are exchanged by the duality for homogeneous algebras of degree N (N-homogeneous algebras). These are the direct extension to arbitrary N of the concepts defined for quadratic algebras ($N = 2$), [26], [27], [25] and our presentation here as well as in Section 3 follows closely the one of reference [27] for quadratic algebras.
In Section 3 we elaborate the categorical setting and we point out the conceptual reason for the occurrence of N-complexes in the framework of N-homogeneous algebras. We also sketch in this section a possible extension of the approach of [27] to quantum spaces and quantum groups in which relations of degree N replace the quadratic ones.
In Section 4 we define the N-complexes which are the generalizations for homogeneous algebras of degree N of the Koszul complexes of quadratic algebras [26], [27]. The definition of the cochain N-complex $L(f)$ associated with a morphism f of N-homogeneous algebras follows immediately from the structure of the unit object $\wedge_N\{d\}$ of one of the (tensor) products of N-homogeneous algebras. We give three equivalent definitions of the chain N-complexe $K(f)$: A first one by dualization of the definition of $L(f)$, a

second one which is an adaptation of [25] by using Lemma 1, and a third one which is a component-wise approach. It is pointed out in this section that one cannot generalize naively the notion of Koszulity for N-homogeneous algebras with $N \geq 3$ by the acyclicity of the appropriate Koszul N-complexes. In Section 5, we recall the definition of Koszul homogeneous algebras of [8] as well as some results of [8], [9] which justify this definition. It is then shown that this definition of Koszulity for homogeneous N-algebras is optimal within the framework of the appropriate Koszul N-complex.

Let us give some indications on our notations. Throughout the paper the symbol $\otimes$ denotes the tensor product over the basic field $\mathbb{K}$. Concerning the generalized homology of N-complexes we shall use the notation of [20] which is better adapted than other ones to the case of chain N-complexes, that is if $E = \oplus_n E_n$ is a chain N-complex with N-differential d, its generalized homology is denoted by ${}_pH(E) = \oplus_{n\in\mathbb{Z}}\ {}_pH_n(E)$ with

$$ {}_pH_n(E) = \mathrm{Ker}(d^p : E_n \to E_{n-p})/\mathrm{Im}(d^{N-p} : E_{n+N-p} \to E_n) $$

for $p \in \{1, \dots, N-1\}$, $(n \in \mathbb{Z})$.

2 Homogeneous algebras of degree N

Let N be an integer with $N \geq 2$. A *homogeneous algebra of degree N* or *N-homogeneous algebra* is an algebra of the form

$$ \mathcal{A} = A(E, R) = T(E)/(R) \tag{1} $$

where E is a finite-dimensional vector space (over $\mathbb{K}$), $T(E)$ is the tensor algebra of E and (R) is the two-sided ideal of $T(E)$ generated by a linear subspace R of $E^{\otimes N}$. The homogeneity of (R) implies that $\mathcal{A}$ is a graded algebra $\mathcal{A} =$

$\oplus_{n\in\mathbb{N}}\mathcal{A}_n$ with $\mathcal{A}_n = E^{\otimes^n}$ for $n < N$ and $\mathcal{A}_n = E^{\otimes^n} / \sum_{r+s=n-N} E^{\otimes^r} \otimes R \otimes E^{\otimes^s}$ for $n \geq N$ where we have set $E^{\otimes^0} = \mathbb{K}$ as usual. Thus $\mathcal{A}$ is a graded algebra which is connected ($\mathcal{A}_0 = \mathbb{K}$), generated in degree 1 ($\mathcal{A}_1 = E$) with the ideal of relations among the elements of $\mathcal{A}_1 = E$ generated by $R \subset E^{\otimes^N} = (\mathcal{A}_1)^{\otimes^N}$.

A *morphism of N-homogeneous algebras* $f : A(E,R) \to A(E',R')$ is a linear mapping $f : E \to E'$ such that $f^{\otimes^N}(R) \subset R'$. Such a morphism is a homomorphism of unital graded algebras. Thus one has a category $\mathbf{H}_N\mathbf{Alg}$ of N-homogeneous algebras and the forgetful functor $\mathbf{H}_N\mathbf{Alg} \to \mathbf{Vect}$, $\mathcal{A} \mapsto E$, from $\mathbf{H}_N\mathbf{Alg}$ to the category $\mathbf{Vect}$ of finite-dimensional vector spaces (over $\mathbb{K}$).

Let $\mathcal{A} = A(E,R)$ be a N-homogeneous algebra. One defines *its dual* $\mathcal{A}^!$ to be the N-homogeneous algebra $\mathcal{A}^! = A(E^*, R^\perp)$ where E^* is the dual vector space of E and where $R^\perp \subset E^{*\otimes^N} = (E^{\otimes^N})^*$ is the annihilator of R i.e. the subspace $\{\omega \in (E^{\otimes^N})^* | \omega(x) = 0, \;\; \forall x \in R\}$ of $(E^{\otimes^N})^*$ identified with $E^{*\otimes^N}$. One has canonically

$$(\mathcal{A}^!)^! = \mathcal{A} \tag{2}$$

and if $f : \mathcal{A} \to \mathcal{A}' = A(E',R')$, is a morphism of $\mathbf{H}_N\mathbf{Alg}$, the transposed of $f : E \to E'$ is a linear mapping of E'^* into E^* which induces the morphism $f^! : (\mathcal{A}')^! \to \mathcal{A}^!$ of $\mathbf{H}_N\mathbf{Alg}$ so ($\mathcal{A} \mapsto \mathcal{A}^!$, $f \mapsto f^!$) is a contravariant (involutive) functor.

Let $\mathcal{A} = A(E,R)$ and $\mathcal{A}' = A(E',R')$ be N-homogeneous algebras; one defines $\mathcal{A} \circ \mathcal{A}'$ and $\mathcal{A} \bullet \mathcal{A}'$ by setting

$$\mathcal{A} \circ \mathcal{A}' = A(E \otimes E', \;\; \pi_N(R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'))$$

$$\mathcal{A} \bullet \mathcal{A}' = A(E \otimes E', \;\; \pi_N(R \otimes R'))$$

where π_N is the permutation

$$(1, 2, \ldots, 2N) \mapsto (1, N+1, 2, N+2, \ldots, k, N+k, \ldots, N, 2N) \qquad (3)$$

belonging to the symmetric group S_{2N} acting as usually on the factors of the tensor products. One has canonically

$$(\mathcal{A} \circ \mathcal{A}')^! = \mathcal{A}^! \bullet \mathcal{A}'^!, \quad (\mathcal{A} \bullet \mathcal{A}')^! = \mathcal{A}^! \circ \mathcal{A}'^! \qquad (4)$$

which follows from the identity $\{R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'\}^\perp = R^\perp \otimes R'^\perp$. On the other hand the inclusion $R \otimes R' \subset R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'$ induces an surjective algebra-homomorphism $p : \mathcal{A} \bullet \mathcal{A}' \to \mathcal{A} \circ \mathcal{A}'$ which is of course a morphism of $\mathbf{H}_N\mathbf{Alg}$.

It is worth noticing here that in contrast with what happens for quadratic algebras if $\mathcal{A}$ and $\mathcal{A}'$ are homogeneous algebras of degree N with $N \geq 3$ then the tensor product algebra $\mathcal{A} \otimes \mathcal{A}'$ is no more a N-homogeneous algebra. Nevertheless there still exists an injective homomorphism of unital algebra $i : \mathcal{A} \circ \mathcal{A}' \to \mathcal{A} \otimes \mathcal{A}'$ doubling the degree which we now describe. Let $\tilde{\imath} : T(E \otimes E') \to T(E) \otimes T(E')$ be the injective linear mapping which restricts as

$$\tilde{\imath} = \pi_n^{-1} : (E \otimes E')^{\otimes^n} \to E^{\otimes^n} \otimes E'^{\otimes^n}$$

on $T^n(E \otimes E') = (E \otimes E')^{\otimes^n}$ for any $n \in \mathbb{N}$. It is straightforward that $\tilde{\imath}$ is an algebra-homomorphism which is an isomorphism onto the subalgebra $\oplus_n E^{\otimes^n} \otimes E'^{\otimes^n}$ of $T(E) \otimes T(E')$. The following proposition is not hard to verify.

PROPOSITION 1 *Let $\mathcal{A} = A(E, R)$ and $\mathcal{A}' = A(E', R')$ be two N-homogeneous algebras. Then $\tilde{\imath}$ passes to the quotient and induces an injective homomorphism i of unital algebras of $\mathcal{A} \circ \mathcal{A}'$ into $\mathcal{A} \otimes \mathcal{A}'$. The image of i is the subalgebra $\oplus_n \mathcal{A}_n \otimes \mathcal{A}'_n$ of $\mathcal{A} \otimes \mathcal{A}'$.*

The proof is almost the same as for quadratic algebras [27].

Remark. As pointed out in [27], any finitely related and finitely generated graded algebra (so in particular any N-homogeneous algebra) gives rise to a quadratic algebra. Indeed if $\mathcal{A} = \oplus_{n\geq 0}\mathcal{A}_n$ is a graded algebra, define $\mathcal{A}^{(d)}$ by setting $\mathcal{A}^{(d)} = \oplus_{n\geq 0}\mathcal{A}_{nd}$. Then it was shown in [3] that if $\mathcal{A}$ is generated by the finite-dimensional subspace $\mathcal{A}_1$ of its elements of degree 1 with the ideal of relations generated by its components of degree $\leq r$, then the same is true for $\mathcal{A}^{(d)}$ with r replaced by $2+(r-2)/d$.

3 Categorical properties

Our aim in this section is to investigate the properties of the category $\mathbf{H}_N\mathbf{Alg}$. We follow again closely [27] replacing the quadratic algebras considered there by the N-homogeneous algebras.

Let $\mathcal{A} = A(E,R)$, $\mathcal{A}' = A(E',R')$ and $\mathcal{A}'' = A(E'',R'')$ be three homogeneous algebras of degree N. Then the isomorphisms $E\otimes E' \simeq E'\otimes E$ and $(E\otimes E')\otimes E'' \simeq E\otimes(E'\otimes E'')$ of **Vect** induce corresponding isomorphisms $\mathcal{A}\circ\mathcal{A}' \simeq \mathcal{A}'\circ\mathcal{A}$ and $(\mathcal{A}\circ\mathcal{A}')\circ\mathcal{A}'' \simeq \mathcal{A}\circ(\mathcal{A}'\circ\mathcal{A}'')$ of N-homogeneous algebras (i.e. of $\mathbf{H}_N\mathbf{Alg}$). Thus $\mathbf{H}_N\mathbf{Alg}$ endowed with $\circ$ is a tensor category [10] and furthermore to the 1-dimensional vector space $\mathbb{K}t\in$ **Vect** which is a unit object of $(\mathbf{Vect},\otimes)$ corresponds the polynomial algebra $\mathbb{K}[t] = A(\mathbb{K}t,0)\simeq T(\mathbb{K})$ as unit object of $(\mathbf{H}_N\mathbf{Alg},\circ)$. In fact the isomorphisms $\mathbb{K}[t]\circ\mathcal{A}\simeq\mathcal{A}\simeq\mathcal{A}\circ\mathbb{K}[t]$ are obvious in $\mathbf{H}_N\mathbf{Alg}$. Thus one has Part (i) of the following theorem.

THEOREM 1 *The category* $\mathbf{H}_N\mathbf{Alg}$ *of N-homogeneous algebras has the* *following properties* (i) *and* (ii)

(i) $\mathbf{H}_N\mathbf{Alg}$ *endowed with* $\circ$ *is a tensor category with unit object* $\mathbb{K}[t]$.

(ii) $\mathbf{H}_N\mathbf{Alg}$ *endowed with* $\bullet$ *is a tensor category with unit object* $\wedge_N\{d\} = \mathbb{K}[t]^!$.

Part (ii) follows from (i) by the duality $\mathcal{A} \mapsto \mathcal{A}^!$. In fact (i) and (ii) are equivalent in view of (2) and (4).

The N-homogeneous algebra $\wedge_N\{d\} = \mathbb{K}[t]^! \simeq T(\mathbb{K})/\mathbb{K}^{\otimes^N}$ is the (unital) graded algebra generated in degree one by d with relation $d^N = 0$. Part (ii) of Theorem 1 is the very reason for the appearance of N-complexes in the present context, remembering the obvious fact that graded $\wedge_N\{d\}$-module and N-complexe are the same thing.

THEOREM 2 *The functorial isomorphism in* **Vect**

$$Hom_{\mathbb{K}}(E \otimes E', E'') = Hom_{\mathbb{K}}(E, E'^* \otimes E'')$$

induces a corresponding functorial isomorphism

$$Hom(\mathcal{A} \bullet \mathcal{B}, \mathcal{C}) = Hom(\mathcal{A}, \mathcal{B}^! \circ \mathcal{C})$$

in $\mathbf{H}_N\mathbf{Alg}$, *(setting* $\mathcal{A} = A(E,R)$, $\mathcal{B} = A(E',R')$ *and* $\mathcal{C} = A(E'',R'')$*).*

Again the proof is the same as for quadratic algebras [27]. It follows that the tensor category $(\mathbf{H}_N\mathbf{Alg}, \bullet)$ has an internal **Hom** [10] given by

$$\mathbf{Hom}(\mathcal{B}, \mathcal{C}) = \mathcal{B}^! \circ \mathcal{C} \tag{5}$$

for two N-homogeneous algebras $\mathcal{B}$ and $\mathcal{C}$. Setting $\mathcal{A} = A(E,R)$, $\mathcal{B} = A(E',R')$ and $\mathcal{C} = A(E'',R'')$ one verifies that the canonical linear mappings $(E^* \otimes E') \otimes E \to E'$ and $(E'^* \otimes E'') \otimes (E^* \otimes E') \to E^* \otimes E''$ induce products

$$\mu : \mathbf{Hom}(\mathcal{A}, \mathcal{B}) \bullet \mathcal{A} \to \mathcal{B} \tag{6}$$

$$m : \mathbf{Hom}(\mathcal{B},\mathcal{C}) \bullet \mathbf{Hom}(\mathcal{A},\mathcal{B}) \to \mathbf{Hom}(\mathcal{A},\mathcal{C}) \tag{7}$$

these internal products as well as their associativity properties follow more generally from the formalism of tensor categories [10].

Following [27], define $\mathbf{hom}(\mathcal{A},\mathcal{B}) = \mathbf{Hom}(\mathcal{A}^!,\mathcal{B}^!)^! = \mathcal{A}^! \bullet \mathcal{B}$. Then one obtains by duality from (6) and (7) morphisms

$$\delta_\circ : \mathcal{B} \to \mathbf{hom}(\mathcal{A},\mathcal{B}) \circ \mathcal{A} \tag{8}$$

$$\Delta_\circ : \mathbf{hom}(\mathcal{A},\mathcal{C}) \to \mathbf{hom}(\mathcal{B},\mathcal{C}) \circ \mathbf{hom}(\mathcal{A},\mathcal{B}) \tag{9}$$

satisfying the corresponding coassociativity properties from which one obtains by composition with the corresponding homomorphisms i the algebra homomorphisms

$$\delta : \mathcal{B} \to \mathbf{hom}(\mathcal{A},\mathcal{B}) \otimes \mathcal{A} \tag{10}$$

$$\Delta : \mathbf{hom}(\mathcal{A},\mathcal{C}) \to \mathbf{hom}(\mathcal{B},\mathcal{C}) \otimes \mathbf{hom}(\mathcal{A},\mathcal{B}) \tag{11}$$

THEOREM 3 *Let $\mathcal{A} = A(E,R)$ be a N-homogeneous algebra. Then the (N-homogeneous) algebra* $\mathbf{end}(\mathcal{A}) = \mathcal{A}^! \bullet \mathcal{A} = \mathbf{hom}(\mathcal{A},\mathcal{A})$ *endowed with the coproduct Δ becomes a bialgebra with counit $\varepsilon : \mathcal{A}^! \bullet \mathcal{A} \to \mathbb{K}$ induced by the duality $\varepsilon = \langle\cdot,\cdot\rangle : E^* \otimes E \to \mathbb{K}$ and δ defines on $\mathcal{A}$ a structure of left* $\mathbf{end}(\mathcal{A})$*-comodule.*

4 The N-complexes $L(f)$ and $K(f)$

Let us apply Theorem 2 with $\mathcal{A} = \wedge_N\{d\}$ and use Theorem 1 (ii). One has

$$\mathrm{Hom}(\mathcal{B},\mathcal{C}) = \mathrm{Hom}(\wedge_N\{d\}, \mathcal{B}^! \circ \mathcal{C}) \tag{12}$$

and we denote by $\xi_f \in \mathcal{B}^! \circ \mathcal{C}$ the image of d corresponding to the morphism $f \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$. One has $(\xi_f)^N = 0$ and by using the injective algebra-homomorphism $i : \mathcal{B}^! \circ \mathcal{C} \to \mathcal{B}^! \otimes \mathcal{C}$ of Proposition 1 we let d be the left multiplication by $i(\xi_f)$ in $\mathcal{B}^! \otimes \mathcal{C}$. One has $d^N = 0$ so, equipped with the appropriate graduation, $(\mathcal{B}^! \otimes \mathcal{C}, d)$ is a N-complex which will be denoted by $L(f)$. In the case where $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and where f is the identity mapping $I_{\mathcal{A}}$ of $\mathcal{A}$ onto itself, this N-complex will be denoted by $L(\mathcal{A})$. These N-complexes are the generalizations of the Koszul complexes denoted by the same symbols for quadratic algebras and morphisms [27]. Note that $(\mathcal{B}^! \otimes \mathcal{C}, d)$ is a cochain N-complex of right $\mathcal{C}$-modules, i.e. $d : \mathcal{B}^!_n \otimes \mathcal{C} \to \mathcal{B}^!_{n+1} \otimes \mathcal{C}$ is $\mathcal{C}$-linear.

Similarly the Koszul complexes $K(f)$ associated with morphisms f of quadratic algebras generalize as N-complexes for morphisms of N-homogeneous algebras. Let $\mathcal{B} = A(E,R)$ and $\mathcal{C} = A(E',R')$ be two N-homogeneous algebras and let $f : \mathcal{B} \to \mathcal{C}$ be a morphism of N-homogeneous algebras ($f \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$). One can define the N-complex $K(f) = (\mathcal{C} \otimes \mathcal{B}^{!*}, d)$ by using partial dualization of the N-complex $L(f)$ generalizing thereby the construction of [26] or one can define $K(f)$ by generalizing the construction of [27], [25].

The first way consists in applying the functor $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{C})$ to each right $\mathcal{C}$-module of the N-complex $(\mathcal{B}^! \otimes \mathcal{C}, d)$. We get a chain N-complex of left $\mathcal{C}$-modules. Since $\mathcal{B}^!_n$ is a finite-dimensional vector space, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}^!_n \otimes \mathcal{C},\mathcal{C})$ is canonically identified to the left module $\mathcal{C} \otimes (\mathcal{B}^!_n)^*$. Then we get the N-complex $K(f)$ whose differential d is easily described in terms of f. In the case $\mathcal{A} = \mathcal{B} = \mathcal{C}$ and $f = I_{\mathcal{A}}$, this complex will be denoted by $K(\mathcal{A})$.

We shall follow hereafter the second more explicit way. Let us associate with $f \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$ the homogeneous linear mapping of degree zero $\alpha : (\mathcal{B}^!)^* \to \mathcal{C}$

defined by setting $\alpha = f : E \to E'$ in degree 1 and $\alpha = 0$ in degrees different from 1. The dual $(\mathcal{B}^!)^*$ of $\mathcal{B}^!$ defined degree by degree is a graded coassociative counital coalgebra and one has $\alpha^{*N} = \underbrace{\alpha * \cdots * \alpha}_{N} = 0$. Indeed it follows from the definition that α^{*N} is trivial in degrees $n \neq N$. On the other hand in degree N, α^{*N} is the composition

$$R \stackrel{f^{\otimes^N}}{\longrightarrow} E'^{\otimes^N} \longrightarrow E'^{\otimes^N}/R'$$

which vanishes since $f^{\otimes^N}(R) \subset R'$. Applying Lemma 1 it is easily checked that the N-differential

$$d_\alpha : \mathcal{C} \otimes \mathcal{B}^{!*} \to \mathcal{C} \otimes \mathcal{B}^{!*}$$

coincides with d of the first way.

Let us give an even more explicit description of $K(f)$ and pay some attention to the degrees. Recall that by $(\mathcal{B}^!)^*$ we just mean here the direct sum $\oplus_n (\mathcal{B}^!_n)^*$ of the dual spaces $(\mathcal{B}^!_n)^*$ of the finite-dimensional vector spaces $\mathcal{B}^!_n$. On the other hand, with $\mathcal{B} = A(E,R)$ as above, one has

$$\mathcal{B}^!_n = E^{*\otimes^n} \quad \text{if} \quad n < N$$

and

$$\mathcal{B}^!_n = E^{*\otimes^n} / \sum_{r+s=n-N} E^{*\otimes^r} \otimes R^\perp \otimes E^{*\otimes^s} \ \text{ if } \ n \geq N.$$

So one has for the dual spaces

$$(\mathcal{B}^!_n)^* \cong E^{\otimes^n} \quad \text{if} \quad n < N \tag{13}$$

and

$$(\mathcal{B}^!_n)^* \cong \bigcap_{r+s=n-N} E^{\otimes^r} \otimes R \otimes E^{\otimes^s} \ \text{ if } \ n \geq N. \tag{14}$$

In view of (13) and (14), one has canonical injections

$$(\mathcal{B}^!_n)^* \hookrightarrow (\mathcal{B}^!_k)^* \otimes (\mathcal{B}^!_\ell)^*$$

for $k+\ell=n$ and one sees that the coproduct Δ of $(\mathcal{B}^!)^*$ is given by

$$\Delta(x) = \sum_{k+\ell=n} x_{k\ell}$$

for $x \in (\mathcal{B}^!_n)^*$ where the $x_{k\ell}$ are the images of x into $(\mathcal{B}^!_k)^* \otimes (\mathcal{B}^!_\ell)^*$ under the above canonical injections.

If $f : \mathcal{B} \to \mathcal{C} = A(E', R')$ is a morphism of $\mathbf{H}_N\mathbf{Alg}$, one verifies that the N-differential d of $K(f)$ defined above is induced by the linear mappings

$$c \otimes (e_1 \otimes e_2 \otimes \cdots \otimes e_n) \mapsto cf(e_1) \otimes (e_2 \otimes \cdots \otimes e_n) \tag{15}$$

of $\mathcal{C} \otimes E^{\otimes^n}$ into $\mathcal{C} \otimes E^{\otimes^{n-1}}$. One has $d\big(\mathcal{C}_s \otimes (\mathcal{B}^!_r)^*\big) \subset \mathcal{C}_{s+1} \otimes (\mathcal{B}^!_{r-1})^*$ so the N-complex $K(f)$ splits into subcomplexes

$$K(f)^n = \oplus_m \mathcal{C}_{n-m} \otimes (\mathcal{B}^!_m)^*, \quad n \in \mathbb{N}$$

which are homogeneous for the total degree. Using (13), (14), (15) one can describe $K(f)^0$ as

$$\cdots \to 0 \to \mathbb{K} \to 0 \to \cdots \tag{16}$$

and $K(f)^n$ as

$$\cdots \to 0 \to E^{\otimes^n} \stackrel{f\otimes I_E^{\otimes^{n-1}}}{\longrightarrow} E' \otimes E^{\otimes^{n-1}} \to \cdots \stackrel{I_{E'}^{\otimes^{n-1}}\otimes f}{\longrightarrow} E'^{\otimes^n} \to 0 \to \cdots \tag{17}$$

for $1 \leq n \leq N-1$ while $K(f)^N$ reads

$$\cdots 0 \to R \stackrel{f\otimes I_E^{\otimes^{N-1}}}{\longrightarrow} E' \otimes E^{\otimes^{N-1}} \to \cdots \to E'^{\otimes^{N-1}} \otimes E \stackrel{can}{\to} \mathcal{C}_N \to 0 \cdots \tag{18}$$

where *can* is the composition of $I_{E'}^{\otimes^{N-1}} \otimes f$ with canonical projection of $E'^{\otimes^N}$ onto $E'^{\otimes^N}/R' = \mathcal{C}_N$.

Let us seek for conditions of maximal acyclicity for the N-complex $K(f)$. Firstly, it is clear that $K(f)^0$ is not acyclic, one has ${}_pH_0(K(f)^0) = \mathbb{K}$ for $p \in \{1, \dots, N-1\}$. Secondly if $N \geq 3$, it is straightforward that if $n \in \{1, \dots, N-2\}$ then $K(f)^n$ is acyclic if and only if $E = E' = 0$. Next comes the following lemma.

LEMMA 2 *The N-complexes $K(f)^{N-1}$ and $K(f)^N$ are acyclic if and only if f is an isomorphism of N-homogeneous algebras.*

Proof. First $K(f)^{N-1}$ is acyclic if and only if f induces an isomorphism $f : E \stackrel{\simeq}{\to} E'$ of vector spaces as easily verified and then, the acyclicity of $K(f)^N$ is equivalent to $f^{\otimes^N}(R) = R'$ which means that f is an isomorphism of N-homogeneous algebras.□

It is worth noticing here that for $N \geq 3$ the nonacyclicity of the $K(f)^n$ for $n \in \{1, \dots, N-2\}$ whenever E or E' is nontrivial is easy to understand and to possibly cure. Let us assume that $K(f)^{N-1}$ and $K(f)^N$ are acyclic. Then by identifying through the isomorphism f the two N-homogeneous algebras, one can assume that $\mathcal{B} = \mathcal{C} = \mathcal{A} = A(E, R)$ and that f is the identity mapping $I_{\mathcal{A}}$ of $\mathcal{A}$ onto itself, that is with the previous notation that one is dealing with $K(f) = K(\mathcal{A})$. Trying to make $K(\mathcal{A})$ as acyclic as possible one is now faced to the following result for $N \geq 3$.

PROPOSITION 2 *Assume that $N \geq 3$, then one has*

$$\mathrm{Ker}(d^{N-1} : \mathcal{A}_2 \otimes (\mathcal{A}^!_{N-1})^* \to \mathcal{A}_{N+1}) = \mathrm{Im}(d : \mathcal{A}_1 \otimes (\mathcal{A}^!_N)^* \to \mathcal{A}_2 \otimes (\mathcal{A}^!_{N-1})^*)$$

if and only if either $R = E^{\otimes^N}$ or $R = 0$.

Proof. One has

$$\mathcal{A}_2 \otimes (\mathcal{A}^!_{N-1})^* = E^{\otimes^2} \otimes E^{\otimes^{N-1}} \simeq E^{\otimes^{N+1}}, \mathcal{A}_{N+1} \simeq E^{\otimes^{N+1}}/E \otimes R + R \otimes E$$

and d^{N-1} identifies here with the canonical projection

$$E^{\otimes^{N+1}} \to E^{\otimes^{N+1}}/E \otimes R + R \otimes E$$

so its kernel is $E\otimes R+R\otimes E$. On the other hand one has $\mathcal{A}_1 \otimes (\mathcal{A}^!_N)^* = E\otimes R$ and $d : E \otimes R \to E^{\otimes^{N+1}}$ is the inclusion. So $\mathrm{Im}(d) = \mathrm{Ker}(d^{N-1})$ is here equivalent to $R \otimes E = E \otimes R + R \otimes E$ and thus to $R \otimes E = E \otimes R$ since all vector spaces are finite-dimensional. It turns out that this holds if and only if either $R = E^{\otimes^N}$ or $R = 0$ (see the appendix). $\square$

COROLLARY 1 *Assume that $N \geq 3$ and let $\mathcal{A} = A(E, R)$ be a N-homogeneous algebra. Then the $K(\mathcal{A})^n$ are acyclic for $n \geq N - 1$ if and only if either $R = 0$ or $R = E^{\otimes^N}$.*

Proof. In view of Proposition 2, $R = 0$ or $R = E^{\otimes^N}$ is necessary for the acyclicity of $K(\mathcal{A})^{N+1}$; on the other hand if $R = 0$ or $R = E^{\otimes^N}$ then the acyclicity of the $K(\mathcal{A})^n$ for $n \geq N - 1$ is obvious. $\square$

Notice that $R = 0$ means that $\mathcal{A}$ is the tensor algebra $T(E)$ whereas $R = E^{\otimes^N}$ means that $\mathcal{A} = T(E^*)^!$. Thus the acyclicity of the $K(\mathcal{A})^n$ for $n \geq N-1$ is stable by the duality $\mathcal{A} \mapsto \mathcal{A}^!$ as for quadratic algebras ($N = 2$). However for $N \geq 3$ this condition does not lead to an interesting class of algebras contrary to what happens for $N = 2$ where it characterizes the Koszul algebras [29]. This is the very reason why another generalization of Koszulity has been introduced and studied in [8] for N-homogeneous algebras.

5 Koszul homogeneous algebras

Let us examine more closely the N-complex $K(\mathcal{A})$:

$$\cdots \longrightarrow \mathcal{A} \otimes (\mathcal{A}^!_i)^* \stackrel{d}{\longrightarrow} \mathcal{A} \otimes (\mathcal{A}^!_{i-1})^* \longrightarrow \cdots \longrightarrow \mathcal{A} \otimes (\mathcal{A}^!_1)^* \stackrel{d}{\longrightarrow} \mathcal{A} \longrightarrow 0\,.$$

The $\mathcal{A}$-linear map $d : \mathcal{A} \otimes (\mathcal{A}^!_i)^* \to \mathcal{A} \otimes (\mathcal{A}^!_{i-1})^*$ is induced by the canonical injection (see in last section)

$$(\mathcal{A}^!_i)^* \hookrightarrow (\mathcal{A}^!_1)^* \otimes (\mathcal{A}^!_{i-1})^* = \mathcal{A}_1 \otimes (\mathcal{A}^!_{i-1})^* \subset \mathcal{A} \otimes (\mathcal{A}^!_{i-1})^*.$$

The degree i of $K(\mathcal{A})$ as N-complex has not to be confused with the total degree n. Recall that, when $N = 2$, the quadratic algebra $\mathcal{A}$ is said to be Koszul if $K(\mathcal{A})$ is acyclic at any degree $i > 0$ (clearly it is equivalent to saying that each complex $K(\mathcal{A})^n$ is acyclic for any total degree $n > 0$).

For any N, it is possible to contract the N-complex $K(\mathcal{A})$ into (2-)complexes by putting together alternately p or $N-p$ arrows d in $K(\mathcal{A})$. The complexes so obtained are the following ones

$$\cdots \xrightarrow{d^{N-p}} \mathcal{A} \otimes (\mathcal{A}^!_{N+r})^* \xrightarrow{d^p} \mathcal{A} \otimes (\mathcal{A}^!_{N-p+r})^* \xrightarrow{d^{N-p}} \mathcal{A} \otimes (\mathcal{A}^!_r)^* \xrightarrow{d^p} 0\,,$$

which are denoted by $C_{p,r}$. All the possibilities are covered by the conditions $0 \leq r \leq N-2$ and $r+1 \leq p \leq N-1$. Note that the complex $C_{p,r}$ at degree i is $\mathcal{A} \otimes (\mathcal{A}^!_k)^*$, where $k = jN + r$ or $k = (j+1)N - p + r$, according to $i = 2j$ or $i = 2j + 1$ ($j \in \mathbb{N}$).

In [8], the complex $C_{N-1,0}$ is called the *Koszul complex* of $\mathcal{A}$, and the homogeneous algebra $\mathcal{A}$ is said to be *Koszul* if this complex is acyclic at any degree $i > 0$. A motivation for this definition is that Koszul property is equivalent to a purity property of the minimal projective resolution of the trivial module. One has the following result [8], [9] :

PROPOSITION 3 *Let $\mathcal{A}$ be a homogeneous algebra of degree N. For $i = 2j$ or $i = 2j+1$, $j \in \mathbb{N}$, the graded vector space $\mathrm{Tor}^{\mathcal{A}}_i(\mathbb{K}, \mathbb{K})$ lives in degrees $\geq jN$ or $\geq jN + 1$ respectively. Moreover, $\mathcal{A}$ is Koszul if and only if each $\mathrm{Tor}^{\mathcal{A}}_i(\mathbb{K}, \mathbb{K})$ is concentrated in degree jN or $jN + 1$ respectively (purity property).*

When $N = 2$, it is exactly Priddy's definition [29]. Another motivation is that a certain cubic Artin-Schelter regular algebra has the purity property, and this cubic algebra is a good candidate for making non-commutative algebraic geometry [1], [2]. Some other non-trivial examples are contained in [8].

The following result shows how the Koszul complex $C_{N-1,0}$ plays a particular role. Actually all the other contracted complexes of $K(\mathcal{A})$ are irrelevant as far as acyclicity is concerned.

PROPOSITION 4 *Let $\mathcal{A} = A(E, R)$ be a homogeneous algebra of degree $N \geq 3$. Assume that (p, r) is distinct from $(N-1, 0)$ and that $C_{p,r}$ is exact at degree $i = 1$. Then $R = 0$ or $R = E^{\otimes^N}$.*

Proof. Assume $r = 0$, hence $1 \leq p \leq N-2$. Regarding $C_{p,0}$ at degree 1 and total degree $N+1$, one gets the exact sequence

$$E \otimes R \xrightarrow{d^p} E^{\otimes^{N+1}} \xrightarrow{d^{N-p}} E^{\otimes^{N+1}} / E \otimes R + R \otimes E,$$

where the maps are the canonical ones. Thus $E \otimes R = E \otimes R + R \otimes E$, leading to $R \otimes E = E \otimes R$. This holds only if $R = 0$ or $R = E^{\otimes^N}$ (Appendix).

Assume now $1 \leq r \leq N-2$ (hence $r+1 \leq p \leq N-1$). Regarding $C_{p,r}$ at degree 1 and total degree $N+r$, one gets the exact sequence

$$(\mathcal{A}^!_{N+r})^* \xrightarrow{d^p} E^{\otimes^{N+r}} \xrightarrow{d^{N-p}} E^{\otimes^{N+r}} / R \otimes E^{\otimes^r},$$

where the maps are the canonical ones. Thus $(\mathcal{A}^!_{N+r})^* = R \otimes E^{\otimes^r}$, and $R \otimes E^{\otimes^r}$ is contained in $E^{\otimes^r} \otimes R$. So $R \otimes E^{\otimes^r} = E^{\otimes^r} \otimes R$, which implies again $R = 0$ or $R = E^{\otimes^N}$ (Appendix).□

It is easy to check that, if $R = 0$ or $R = E^{\otimes^N}$, any $C_{p,r}$ is exact at any degree $i > 0$. On the other hand, for any R, one has

$$H_0(C_{p,r}) = \oplus_{0\leq j\leq N-p-1}\ E^{\otimes^j} \otimes E^{\otimes^r},$$

which can be considered as a Koszul left $\mathcal{A}$-module if $\mathcal{A}$ is Koszul.

6 Appendix : a lemma on tensor products

LEMMA 3 *Let E be a finite-dimensional vector space. Let R be a subspace of $E^{\otimes^N}$, $N \geq 1$. If $R \otimes E^{\otimes^r} = E^{\otimes^r} \otimes R$ holds for an integer $r \geq 1$, then $R = 0$ or $R = E^{\otimes^N}$.*

Proof. Fix a basis $X = (x_1, \ldots, x_n)$ of E, ordered by $x_1 < \cdots < x_n$. The set X^N of the words of length N in the letters $x_1, \ldots, x_n$ is a basis of $E^{\otimes^N}$ which is lexicographically ordered. Denote by S the X^N-reduction operator of $E^{\otimes^N}$ associated to R [6], [7]. This means the following properties:

(i) S is an endomorphism of the vector space $E^{\otimes^N}$ such that $S^2 = S$,

(ii) for any $a \in X^N$, either $S(a) = a$ or $S(a) < a$ (the latter inequality means $S(a) = 0$, or otherwise any word occuring in the linear decomposition of $S(a)$ on X^N is $< a$ for the lexicographic ordering),

(iii) $\mathrm{Ker}(S) = R$.

Then $S \otimes I_{E^{\otimes^r}}$ and $I_{E^{\otimes^r}} \otimes S$ are the X^{N+r}-reduction operators of $E^{\otimes^{N+r}}$, respectively associated to $R \otimes E^{\otimes^r}$ and $E^{\otimes^r} \otimes R$. By assumption these endomorphisms are equal. In particular, one has

$$\mathrm{Im}(S) \otimes E^{\otimes^r} = E^{\otimes^r} \otimes \mathrm{Im}(S).$$

But the subspace $\mathrm{Im}(S)$ is monomial, i.e. generated by words. So it suffices to prove the lemma when R is monomial.

Assume that R contains the word $x_{i_1} \dots x_{i_N}$. For any letters $x_{j_1}, \dots, x_{j_r}$, the word $x_{i_1} \dots x_{i_N} x_{j_1} \dots x_{j_r}$ belongs to $E^{\otimes^r} \otimes R$. Thus $x_{i_{r+1}} \dots x_{i_N} x_{j_1} \dots x_{j_r}$ belongs to R. Continuing the process, we see that any word belongs to R. $\square$

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